INTERLOCKING DILATIONS

In memory of Maurice Audin

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1. Introduction.

1.1. In this paper we consider the following problem which arose in [2]: given a family of contractions $\{T_\alpha \mid \alpha \varepsilon J\}$ acting in a certain way (see (1) of §1.4 below) on a Hilbert space H does there exist an interlocking minimal dilation $\{U_\alpha \mid \alpha \varepsilon J\}$ acting on some Hilbert space $K \supset H$? We determine necessary and sufficient conditions for the existence of K and $\{U_\alpha\}$ for the case that certain commutativity relations (see (5) of §1.4 below) hold. This result is considerably more general than that obtained in [3].

The precise statement of our result is given in §1.4 below. It will be understood more easily if we first discuss two special cases (i): $\{T_\alpha\}$ consists of a single contraction T and (ii): $\{T_\alpha\}$ consists of two contractions T_1, T_2.

1.2. For a single contraction T we suppose that:

(i) H is assigned a decomposition $H = H_1 \oplus H_2 \oplus H_3$ (written $L \oplus M \oplus R$ in [1]) such that

$$TH_1 \subset H_1 \oplus H_2 \quad \text{and} \quad T(H_2 \oplus H_3) \subset H_3,$$

and we define: $\{U_\alpha\}$, acting on K, is called an *interlocking minimal dilation* of $\{T_\alpha\}$ if:

(ii) $\{U_\alpha\}$ consists of a single unitary operator U,

(iii) K is spanned by the spaces $\{U^n H \mid -\infty < n < \infty\}$,

(iv) For all $x, y \varepsilon H$ and all $n \geq 0$: $(U^n x \mid y) = (T^n x \mid y)$.

Because of (ii) and (iii), U if existing is determined completely on K by (iv). As shown by B. Sz.-Nagy (see [4] or [1]) such K, U exist always (the commutativity relations mentioned in §1.1 do not occur). U is called a *unitary dilation* of T.

As yet the hypothesis (i) has not been used. To motivate later work we let $P_{(u)}$ denote the (orthogonal) projection onto $H_u (u = 1, 2, 3)$ so that $T = TP_{(1)} + T(P_{(2)} + P_{(3)}) = TP_{(1)} + P_{(3)}T$, and $T^n = (TP_{(1)} + P_{(3)}T)^n = \sum_{i=0}^{n} (P_{(3)}T)^{n-i}(TP_{(1)})^i$. Now (iv) can be written as:

$$\text{(iv)}' \qquad (U^n x \mid y) = \sum_{i=0}^{n} ((TP_{(1)})^i x \mid (T^* P_{(3)})^{n-i} y).$$

1.3. For two contractions T_1, T_2 we suppose that

(i) H has been assigned a decomposition $H = \sum_{u,v=1}^{3} \oplus H_{uv}$ such that

$$T_1 H_{1v} \subset H_{1v} \oplus H_{2v} \quad \text{and} \quad T_1(H_{2v} \oplus H_{3v}) \subset H_{3v} \quad \text{for} \quad v = 1, 2, 3,$$

$$T_2 H_{u1} \subset H_{u1} \oplus H_{u2} \quad \text{and} \quad T_2(H_{u2} \oplus H_{u3}) \subset H_{u3} \quad \text{for} \quad u = 1, 2, 3.$$

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