A GENERALIZATION OF STONE'S REPRESENTATION THEOREM FOR BOOLEAN ALGEBRAS

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The well-known theorem of M. H. Stone [10] asserts that every Boolean algebra is isomorphic to a field of sets, that is, to a subalgebra of a complete and atomic Boolean algebra. A generalization of Boolean algebras are the Stone lattices. It is my aim to prove a representation theorem for Stone lattices: *Every Stone lattice is isomorphic to a sublattice of the lattice of all ideals of a complete and atomic Boolean algebra.*

This theorem was conjectured by O. Frink [3] as a solution to Problem 70 of Garrett Birkhoff's book *Lattice Theory* (revised edition, New York 1948), which asks for a characterization of Stone lattices. The name "Stone lattice" was proposed in [5], which contains the first solution to Problem 70. The author's paper [4] contains a further solution, though less deep than the first one. A Stone lattice is a lattice which is distributive and pseudo-complemented, and in which the formula $a^* \cup a^{**} = 1$ holds identically. Stone lattices and the problem of characterizing them were first discussed in Stone's paper [11] on Brouwerian logic.

My proof involves applications of mathematical logic to algebra like those of Jónsson and Tarski, L. Henkin, and A. Robinson [6], [8], [9]. First I omit the assumption that the Boolean algebra is complete, and prove an apparently weaker imbedding theorem; then I show that this weaker theorem is actually equivalent to the original one. This idea is due to Jónsson and Tarski. Next I show that a particular class of Stone lattices forms an arithmetic class in the sense of Tarski. Then by applying a consequence of Gödel's theorem concerning the completeness of the first order functional calculus (see Henkin [6] and Hintikka [7]), we conclude that it is sufficient to consider finitely generated Stone lattices, which are finite. A direct decomposition theorem further simplifies the problem; the resulting class of Stone lattices is so simple that a representation is obtained without difficulty.

1. **Stone lattices.** A Stone lattice is a set of elements L in which operations $\cup$, $\cap$ and * are defined, the first two being binary, and the third unary; two elements 0 and 1 of L are fixed. The following axioms are assumed:

S_1 $\quad x \cup y = y \cup x.$

S_2 $\quad x \cap y = y \cap x.$

S_3 $\quad x \cup (y \cup z) = (x \cup y) \cup z.$

S_4 $\quad x \cap (y \cap z) = (x \cap y) \cap z.$

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