OPEN FUNCTIONS AND DIMENSION

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1. Introduction and known results. Suppose that X and Y are metric spaces and f is an open function from X onto Y. The purpose of this paper is to study the relation between dim X and dim Y, where dim denotes the covering dimension [5; 5] of a space X. Let us quickly review the known dimension theoretic results concerning open functions. Alexandroff [4; 68] proved that if X is a compact metric space, Y a metric space, and f an open, continuous, at most countable-to-one function from X onto Y, then dim X = dim Y. Roberts [8] improved on this result in one direction by showing that if X is a separable metric space, Y a locally compact separable metric space, and f an open function from X onto Y such that for each point y in Y, $f^{-1}(y)$ is not dense in itself, then dim $Y \leq$ dim X. We shall give generalizations and modifications of these results. Theorem 3.2 gives a modification of Roberts' Theorem; Theorems 4.1, 4.2, 4.3, and 4.4 give modifications of Alexandroff's Theorem; Theorem 5.1 gives a generalization of Alexandroff's Theorem.

Let X and Y be topological spaces, and let f be a function from X onto Y. The function f is said to be *open* (*closed*) if for each open (closed) subset U of X, $f(U)$ is an open (closed) subset of Y; the function f is said to be *locally one-to-one* if for each point p in X, there is a neighborhood U of p such that the function $(f \mid U)$ is one-to-one. A metric space X is said to be *locally separable* if given any point p in X, there is a neighborhood U of p which is separable. All spaces in this paper are assumed to be at least Hausdorff.

2. Open functions which do not lower dimension.

2.1. LEMMA. *Let X and Y be topological spaces, let f be an open function from X onto Y, let U be an open subset of X, and let $V = \{x : x \text{ in } U;\ (f \mid U) \text{ is one-to-one}\}$. Then $f(V)$ is closed in $f(U)$. Moreover, if f is continuous, then V is closed in U.*

Proof. Let y be a point in $f(U)$ such that y is not in $f(V)$. We shall show that y is not a limit point of $f(V)$. Let x and x' be distinct points in U such that $f(x) = f(x') = y$. Let W and W' be disjoint open neighborhoods of x and x' respectively which are contained in U. Then $Z(Z = f(W) \cap f(W'))$ is an open neighborhood of y, and $Z \cap f(V) = \phi$.

Now suppose that f is continuous also, and let x be a point in U which is a

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