A NECESSARY AND SUFFICIENT CONDITION FOR A DISCRETE GROUP OF LINEAR FRACTIONAL TRANSFORMATIONS TO BE DISCONTINUOUS

By H. Larcher

In the following G denotes a group of linear fractional transformations. The elements of G we denote by $s(z)$, $t(z)$, $\cdots$. Then $s(z) = (az + b)/(cz + d)$, where a, b, c, d are finite complex numbers subject to the normalization $ad - bc = 1$. The domain of definition and the range of the $s(z)$ are the Riemann sphere R. A point $z_0 \in R$ is called a limit point of G, if there exists a $z \in R$ and a sequence of infinitely many distinct transformations $\{s_k(z)\}$ of G such that $s_k(z) \to z_0$ as $k \to \infty$. An ordinary point of G is one that is not a limit point. A discontinuous group is one for which there exists an ordinary point.

With every element $s(z) = (az + b)/(cz + d)$ of G we may associate the two matrices $\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The resulting multiplicative group of 2×2 matrices we denote by $\bar{G}$. If $s(z)$ denotes a transformation of G, we use S to denote the corresponding matrix. $\bar{G}$ is said to be discrete if it does not contain a sequence of infinitely many distinct matrices $\{S_k\}$ such that $S_k \to I$, the 2×2 identity matrix. As is customary we call G discrete if $\bar{G}$ is discrete.

The relation between discreteness and discontinuity of a group G has been extensively investigated. It is trivial to see that G is not discontinuous if it is not discrete. On the other hand the example of the Picard group shows that G may be discrete without its being discontinuous [3; 35–36]. In order to delineate the result of this paper from known results we state two theorems which may be found in [2].

(1) Let G be a discrete group of linear fractional transformations. Then G is discontinuous in a domain D (open and connected set in R) if and only if the elements of G form a normal family in D [2; 74].

With regard to normal families of linear fractional transformations we have the theorem.

(2) Let $\{f_r(z)\}$ denote a family of non-singular linear fractional transformations. If there exists a domain D and two points α and β in R such that $f_r(z) \neq \alpha$ and $f_r(z) \neq \beta$ for all elements of the family and for all $z \in D$, then $\{f_r(z)\}$ is a normal family in D [2; 71].

Let GO denote the set of image points of a set O under all transformations of G. The two stated theorems then imply that a discrete group of linear fractional transformations is discontinuous if there exists an open set O, subset of R, and two complex numbers α and β (finite or infinite) such that $\alpha, \beta \notin GO$. This

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