AN ERGODIC APPLICATION OF ALMOST CONVERGENT SEQUENCES

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1. On the space of bounded real sequences $\{x_n\}$ there exist linear functionals L called *Banach limits*, satisfying the conditions

$$L(x_n) \geq 0 \quad \text{if} \quad x_n \geq 0, \qquad n = 0, 1, \cdots; \tag{1}$$

$$L(x_{n+1}) = L(x_n); \tag{2}$$

$$\liminf x_n \leq L(x_n) \leq \limsup x_n; \tag{3}$$

(see [1; 34]; [8; 73]). If there is a number s with $L(x_n) = s$ for all Banach limits L, the sequence $\{x_n\}$ is called *almost convergent*, and one writes: $F\text{-}\lim x_n = s$. It is shown in this note that certain basic ergodic properties: ergodicity and invariance of finite measures, existence of finite invariant equivalent measures, may be simply expressed in terms of almost convergent sequences. A link is thus established with Lorentz's study of almost convergent sequences [16]. We state the main theorem of [16], which will be repeatedly applied in the sequel.

THEOREM 1 (Lorentz). *A sequence $\{x_n\}$ is almost convergent with F-limit s if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=i}^{i+n-1} x_j = s$$

uniformly in i.

Note that certain connections between abstract ergodic theory and Theorem 1 are known; thus Jerison [15; 86] derives Theorem 1 from a Banach space mean ergodic theorem.

2. Let Ω be a non-empty abstract set and let $\mathfrak{A}$ be a σ-field of subsets of Ω. All considered sets will be assumed in $\mathfrak{A}$. A *measurable transformation* T is a mapping from Ω into Ω with $T^{-1}\mathfrak{A} \subset \mathfrak{A}$; if also T^{-1} is a measurable transformation, T is *invertible*. In the sequel, a system $(\Omega, \mathfrak{A}, T)$ will be assumed given, with T measurable, but not necessarily invertible. A *measure* (*α-measure*) p is a countably additive (finitely additive), non-negative set function on $\mathfrak{A}$ with $0 < p(\Omega) < \infty$. (Even when we assume that p is a *probability measure*, i.e. $p(\Omega) = 1$, all results hold, with obvious modifications, if only $p(\Omega) < \infty$.) A set A is *invariant* if $A = T^{-1}A$; a measure p is *ergodic* if $p(A) = 0$ or $p(\Omega - A) = 0$

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