PARTITIONS OF EUCLIDEAN SPACES INTO DENSE, L_n-CONNECTED SETS

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In [1; 110] Hocking and Young construct by transfinite induction two disjoint, dense, connected subsets of E^2 (the plane) whose union is E^2. The construction can be easily modified so as to divide E^2 into α such sets where $\alpha \leq \mathbf{c}$. (By $\mathbf{c}$ is meant the first ordinal equipotent to E^1.) A natural question then arises: for what α, can E^2 (or E^n in general) be partitioned into α disjoint, dense sets which are "connected" in some more restrictive sense such as arcwise-connected or polygonally- connected, etc.? As a partial answer to this, we obtain some results on partitioning $E^k (k > 1)$ and certain of its subsets into disjoint, dense L_n-connected subsets. Our main result is that E^2 can be so partitioned into 2 L_5-connected sets and that neither the "2" nor the "5" can be improved upon.

A subset C of E^k is L_n-connected if each two distinct points in C can be joined by a polygonal arc, having no more than n segments, lying entirely in C. (Our "L_n-connected" is the same as the "L_n-set" as introduced and defined by Horn and Valentine [2].) In particular, if $C(y, z)$ denotes this polygonal arc joining y and z in C, we have $C(y, z) = \bigcup_{i=1}^{m}[x^i, x^{i+1}]$ where $y = x^1$, $z = x^{m+1}$, $m \leq n$ and $x^i \notin [x^{i-1}, x^{i+1}]$ for all j. (For $a, b \in E^k [a, b] = \{\lambda a + (1 - \lambda)b : 0 \leq \lambda \leq 1\}$.) Moreover, if α is any ordinal $\leq \mathbf{c}$, we say $C \subset E^k$ can be *partitioned into α dense, L_n-connected sets* if there exists a family of α disjoint, dense L_n-connected subsets of C whose union is C.

I. Partitions for $\mathbf{E}^2$.

The possibilities for such partitions of E^2 are established by the following three theorems.

THEOREM 1. *E^2 can be partitioned into 2 dense L_5-connected sets.*

Proof. For t a positive real put $T_t = [it, t-it] \cup [t-it, -t-it] \cup [-t-it, it]$ and define $A_1 = \cup\{T_t - \{-it\} : t>0,\ t \text{ rational}\}$ and $A_2 = (\cup\{T_t - \{it\} : t>0,\ t \text{ irrational}\}) \cup \{0\}$. It is easily checked that A_1 and A_2 give the desired partition.

THEOREM 2. *E^2 cannot be partitioned into 2 dense L_n-connected sets for any $n < 5$.*

Proof. Suppose $E^2 = A \cup B$ where A and B are disjoint, dense L_4-connected sets. First note that for $y, z \in A$ the set $A(y, z) = \bigcup_{i=1}^{n} [x^i, x^{i+1}]$, which we call an n-link, will be unique; otherwise we would get a closed curve separating E^2. Using the connectedness and denseness of both A and B, it follows easily

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