# ARCWISE CONNECTED SETS IN THE PLANE

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Our space will be the plane $E$. Let $M = \{\mathrm{Re}^{i\theta} \mid \text{either (1) } \theta = 0, \text{ or (2) } \theta \neq \pi \text{ and } R \text{ is rational}\}$ and let $N = \{\mathrm{Re}^{i\theta} \mid \text{either (1) } \theta = \pi \text{ and } R \neq 0 \text{ or (2) } \theta \neq 0 \text{ and } R \text{ is irrational}\}$. This example of two disjoint dense arcwise connected sets whose union is $E$ has been given by J. G. Ceder who has also raised the question of the existence of *three* disjoint dense arcwise connected sets whose union is $E$.

A subset of $E$ is said to be of category 1 if it is the union of at most countably many nowhere dense sets; a subset of $E$ is of category 2 if it is not of category 1. By the Baire Category Theorem, $E$ is of category 2. Note that the union of countably many sets of category 1 is of category 1.

Theorem. *If there are three disjoint arcwise connected sets each of which is dense in $E$, then each of the sets is of category* 1.

This theorem clearly answers Mr. Ceder's question in the negative; and, in fact, if for some countable cardinal $\alpha > 2$, there are $\alpha$ disjoint arcwise connected dense subsets of $E$, by this theorem each of the sets is of category 1 and hence their union is of category 1.

For the rest of the paper we will assume that $A$, $B$, and $C$ are disjoint arcwise connected dense subsets of $E$ and that $A$ is of category 2. A number of lemmas will be proved in order to reach a contradiction of this assumption.

We will use the letters $n$ and $m$ to stand for positive integers and the letter $i$ to stand for the integers 1 and 2.

For $p$ and $q$ in $A$, let $A(p, q)$ be the unique arc from $p$ to $q$ lying in $A$; also $A(p, p) = \{p\}$. Define $B(p, q)$ and $C(p, q)$ similarly for $p$ and $q$ in $B$ and $C$, respectively.

For $p \in A$, let $G_p = \{X \mid \text{for some } x \in A - \{p\}, X \text{ is the set of all points } y \in A \text{ so that } p \notin A(x, y)\}$. Note that no two elements of $G_p$ have a point in common.

Lemma 1. *If $p \in A$, there is at most one term of $G_p$ of category* 2.

*Proof of Lemma* 1. Suppose that $X_1$ and $X_2$ are two terms of $G_p$ of category 2 and that $x_i \in X_i$.

Since $B$ is dense in $E$, there must be sequences of points of $B$ approaching $p$ from opposite sides of $A(x_1, x_2)$. Therefore we can find arcs $L_1, L_2, \cdots$ such that $L_n \supset L_{n+1}$, $L_1 \cap L_2 \cap \cdots = \{p\}$, the end points of $L_n$ belong to $B$, $L_n \cap A(x_1, x_2) = \{p\}$, and $A(x_1, p)$ and $A(x_2, p)$ abut on $L_n$ from opposite sides.

For each $n$ and $i$, let $X_{in} = \{x \mid A(x, x_i) \cap L_n = 0 \text{ and } x \in X_i\}$. Observe that $X_i = X_{i1} \cup X_{i2} \cup \cdots$; therefore, since $X_i$ is of category 2 and $X_{in} \subset X_{i,n+1}$, there is an $n$ so that both $X_{1n}$ and $X_{2n}$ are of category 2.

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