PERSISTENT MARKOV CHAINS CONSTRUCTED FROM TRANSIENT CHAINS

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1. **Introduction.** A state of a Markov chain is called transient if the probability that the chain never returns to that state is greater than zero. The purpose of this paper is to study the new chain obtained by imposing the conditional hypothesis that the chain must return to the given transient state at some time. We may do this if our chosen state has probability greater than zero of eventual recurrence. In this new chain our chosen state is persistent, as are all states which may be reached from it and from which it may be reached. We have thus constructed a persistent chain from a transient chain, and the problem is to calculate its transition probabilities, which depend on our original chain.

This problem was considered out of purely mathematical interest, but it may serve as a model for describing the fluctuation in the fortune of a successful gambler. Suppose that the gambler begins with a capital of b dollars and has a probability strictly less than one-half of winning one dollar in any one trial. Therefore, with probability one he will eventually lose his fortune. But, if we consider only those gamblers who break even after a long period of time, we have imposed the hypothesis that the gambler's fortune is equal to b at some distant future time. We may then consider the conditional probabilities of the gambler winning x dollars in n trials under the hypothesis his fortune must equal b at some time in the chain. This describes a new, persistent Markov chain which we have constructed from a transient chain.

In this paper I will use standard Markov chain terminology and will specifically deal with a transient chain with the matrix p_{ij}. To discuss Markov chains in terms of recurrent events, we define $f_{ij}^{(n)}$ as the conditional probability that $X_m = j$ (X_m is the random variable X at time m) for some m, given that $X_{m-n} = 1$, that is, $f_{ij}^{(n)} = P(X_m = j, X_{m-1} \neq j, \cdots X_{m-n-1} \neq j \mid X_{m-n} = i)$. We may take generating functions of $f_{ij}^{(n)}$ and $p_{ij}^{(n)}$:

$$F_{ij}(s) = \sum_{n=1}^{\infty} f_{ij}^{(n)} s^n, \qquad P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^{(n)} S^n$$

(and for $i = j$ we assume $f_{ij}^{(0)} = 0$, $p_{ij}^{(0)} = 1$).

Clearly, for one step transitions, the relation $f_{ij}^{(1)} = p_{ij}^{(1)}$ holds and for higher

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