LIMITS OF ENTIRE FUNCTIONS WHOSE GROWTH AND ZEROS ARE RESTRICTED

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1. Introduction and results. A well-known theorem due to Laguerre [5] and Pólya [8] asserts that if $\{P_n\}$ is a sequence of polynomials with only real zeros and $P_n \to f \not\equiv 0$ uniformly on a disc about the origin, then f is (the restriction of) an entire function of the form

$$f(z) = \exp(-az^2 + bz + c) \cdot z^m \prod_n (1 - z/z_n)e^{z/z_n} \tag{1.1}$$

where $a \geq 0$, b and the z_n are real, and $\sum z_n^{-2}$ converges. Conversely, every such entire function can be approximated, uniformly on every bounded set, by polynomials having only real zeros (Pólya [8]).

Several authors have proved extensions of this result to the case where the zeros of the approximating polynomials are confined to sets other than the real axis. References can be found in survey articles by Obrechkoff [7] and the second author [3].

In this note we consider a generalization of the theorem of Laguerre and Pólya in a different direction. We begin by replacing the polynomials with real zeros by real entire functions of less restricted growth whose zeros are real.

For an arbitrary integer $p \geq 0$, let U_p denote the class of holomorphic functions $\not\equiv 0$ which are uniform limits on every bounded set (or just on a disc about the origin) of real entire functions of genus $\leq 2p$ whose zeros are real.

Theorem 1. *The class U_p consists of the (restrictions of the) entire functions of the form*

$$\exp(-az^{2p+2}) \cdot h(z) \tag{1.2}$$

where $a \geq 0$ and h is real entire, of genus $\leq 2p + 1$, and has only real zeros.

The condition that the approximating functions be real on the real axis and have only real zeros may be weakened. If we allow entire functions of restricted growth all of whose zeros lie in a suitably small angular neighborhood of the real axis, we obtain a similar result.

Let S_α denote the pair of angles

$$|\arg z| \leq \alpha, \qquad |\arg z - \pi| \leq \alpha. \tag{1.3}$$

For every integer $p \geq 0$ and for $0 < R_0 \leq \infty$ let $V_p(\alpha, R_0)$ denote the class of holomorphic functions $\not\equiv 0$ which are uniform limits, on every disc $|z| \leq R < R_0$, of entire functions of genus $\leq 2p$ whose zeros lie in S_α.

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