LOCALLY ONE-TO-ONE MAPPINGS AND A CLASSICAL THEOREM ON SCHLICHT FUNCTIONS

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1. **Introduction.** The following classical theorem from the theory of functions of a complex variable (cf. [5], §8.12; 184) has been attributed to Gaston Darboux [6]. For example see [8; 377–381].

Darboux's Theorem. *If $f(z)$ is single-valued and holomorphic in a simply connected open subset R of the complex plane and if $f(z)$ takes no values more than once on some rectifiable simple closed curve C lying in R, then $f(z)$ is schlicht on the compact set X consisting of C and its interior.*

In this paper some topological theorems are given which are similar to Darboux's Theorem and from one of which (Theorem 1) Darboux's Theorem is easily deduced. Our theorems are like Darboux's Theorem in the sense that we assume a certain mapping to be one-to-one on the *boundary* of its domain and conclude that it is one-to-one *everywhere* on its domain.

Theorem 1 (stated and proved in §3) differs from Darboux's Theorem in three respects. In the first place it applies to mappings of compact subsets X of n-dimensional Euclidean space E_n (for $n \geq 2$), and is therefore not restricted to mappings of the complex plane. Next, the requirement of analyticity in Darboux's Theorem is replaced by the requirement that f be locally one-to-one (cf. Definition 1, §2) at each point of its domain X except possibly on a "small" subset Z. The relation of this last property to analyticity is perhaps more clearly understood when one recalls the classical sufficient condition for the local one-to-one-ness, at a point x_0, of a class $\mathcal{C}'$ transformation of E_n; namely, the nonvanishing of the Jacobian determinant at x_0. Indeed, if $f(z) = u(x, y) + iv(x, y)$ is analytic in a connected open subset R of the complex z-plane, the Jacobian determinant of the transformation $f: R \to E_2$ defined by

$$\xi = u(x, y)$$

$$\eta = v(x, y)$$

is precisely $| f'(z) |^2$, and so the exceptional set Z is a subset of the set of zeros of $f'(z)$ in R. It is well known that this last set is discrete, i.e., finite in each compact subset of R. Finally, in Theorem 1 it is not required that the boundary of the schlicht region X be a rectifiable Jordan curve, but merely an irreducible separating set of E_n (cf. Definition 3, §3).

Received December 22, 1961. This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF 49(638)-382, and the Army Research Office (Durham), under Contract Number DA-26-034 Ord.-3220.