SOME ARITHMETIC PROPERTIES OF A SPECIAL SEQUENCE OF POLYNOMIALS IN THE GAUSSIAN FIELD

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To Professor H. S. Vandiver on his eightieth birthday

1. Kelisky [2] has defined a set of rational integers T_n by means of

$$\sum_{n=0}^{\infty} T_n \frac{x^n}{n!} = \exp(\arctan x) \tag{1.1}$$

and proved in particular that

$$T_p \equiv \begin{cases} 0 \pmod{p} & (p = 4n+1) \\ 2 \pmod{p} & (p = 4n+3), \end{cases} \tag{1.2}$$

where p is a prime. More generally the writer [1] has defined the polynomial $T_n(z)$ by means of

$$\sum_{n=0}^{\infty} T_n(z) \frac{x^n}{n!} = \exp(z \arctan x), \tag{1.3}$$

so that $T_n = T_n(1)$, and proved in particular that if $a + bi$ is a number of the Gaussian field that is integral (mod p), where p is a prime, then

$$T_p(a+bi) \equiv \begin{cases} 0 \pmod{p} & (p = 4n+1) \\ 2a \pmod{p} & (p = 4n+3). \end{cases} \tag{1.4}$$

On examining some numerical data Kelisky has conjectured that if $p = 16n^2+1$ is prime, then

$$\frac{1}{p} T_p \equiv -\tfrac{1}{2} \pmod{p}. \tag{1.5}$$

In the present paper we shall determine the residue of $T_p(a+bi)$ (mod p^2), where p is a prime $\equiv 1 \pmod 4$ and $a + bi$ is a number of the Gaussian field that is integral (mod p). In particular we shall show that the conjecture (1.5) is correct. A summary of the results obtained will be found in §7 below.

In addition we shall determine the residue of $T_p(\alpha)$ (mod p^2), where $p \equiv 3$ (mod 4) and α is either real or purely imaginary. We find that

$$T_p(a) \equiv \frac{a}{2^{p-1}} \prod_{r=1}^{\frac{1}{2}(p-1)} (a^2 + 16r^2) + a(2^p - 2) \pmod{p^2}, \tag{1.6}$$

$$T_p(ib) \equiv i\{b + p - 2k - (-1)^k p\} \pmod{p^2}, \tag{1.7}$$

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