LATTICE-EQUIVALENCE OF TOPOLOGICAL SPACES

BY W. J. THRON

1. **Introduction.** Let X be a given set. A topological structure can be defined on X by specifying the family $\mathfrak{C}$ of closed sets in X. The pair $(X, \mathfrak{C})$ is then called a topological space. The family $\mathfrak{C}$ is a complete distributive lattice if set inclusion is taken as the ordering. Two topological spaces $(X, \mathfrak{C})$ and $(Y, \mathfrak{D})$ are said to be homeomorphic iff there exists a 1–1 function f from X onto Y such that $f(C) \in \mathfrak{D}$, for every $C \in \mathfrak{C}$, and $f^{-1}(D) \in \mathfrak{C}$ for every $D \in \mathfrak{D}$. A homeomorphism thus induces a 1–1 map from $\mathfrak{C}$ onto $\mathfrak{D}$, which is order-preserving and has an order-preserving inverse.

We shall call two topological spaces $(X, \mathfrak{C})$ and $(Y, \mathfrak{D})$ lattice-equivalent iff there exists a 1–1 function from $\mathfrak{C}$ onto $\mathfrak{D}$ which together with its inverse is order-preserving. Clearly then homeomorphic spaces are lattice-equivalent. Our primary concern in this paper is to determine what additional conditions have to be imposed on lattice-equivalent spaces in order that they be homeomorphic. This leads us to consider two related questions: First, what abstract complete distributive lattices can be represented as $\mathfrak{C}$-lattices? A necessary and sufficient condition for such a representation is given in §3. Finally we present an analysis of separation axioms by decomposing them into lattice-invariant and other components.

One of the new separation axioms introduced recently by Aull and Thron [1] proved to play an important part in the present context. The author has also profited and been stimulated by the results obtained by some of his students in response to a question on the relation between the lattice structure of $\mathfrak{C}$ and topological properties of the space, which was given as part of a "take-home" examination in a course on point set topology. The most significant results were obtained by H. W. Davis, R. P. Osborne, K. L. Phillips, and A. K. Snyder all of whom proved a weaker form of Theorem 2.1. Our Theorem 2.1 was also independently discovered by J. H. Brooks. Corollary 2.1 was originally proved by Davis and later generalized by the author to the present Theorem 2.3.

When writing the original version of this paper the author was aware only of the results of Stone [6] and Wallman [8] on the representation of an abstract lattice as a $\mathfrak{C}$-lattice for special cases (see also Birkhoff [2; 172–3]), as well as of the book of Vaidyanathaswamy [7] which emphasizes the lattice theoretic aspects of topology, without however over-lapping our results to an appreciable extent. Since then a number of other articles have been brought to the author's attention. In 1960 Kowalski [4] attacked and solved problems very similar to the ones discussed here. However, his methods and the statements of his

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