ALMOST PERIODIC SOLUTIONS AND CRITICAL ROOTS

BY JANE CRONIN

1. **Introduction.** We obtain an estimate of the number of almost periodic solutions of an n-dimensional system of the form

$$\dot{x} = Ax + \mu F(x, t, \mu) + f(t) \tag{E_μ}$$

where A is a constant matrix some of whose characteristic roots are critical (i.e., have zero real parts), functions F and f are almost periodic in t and μ is a small parameter. If A does not have critical roots, then (E_μ) has a unique almost periodic solution for sufficiently small μ. (See Malkin [4, Chapter IV].) If A has m critical roots, the problem reduces to that of finding those solutions of a system of equations

$$P_i(x_1, \cdots, x_m) = 0 \qquad (i = 1, \cdots, m)$$

with the property that the characteristic roots of the matrix $(\partial P_i/\partial x_j)$ have nonzero real parts. So there may be several almost periodic solutions or none. We show that if F is varied arbitrarily slightly (the precise meaning of this term will be explained later) then each solution has this property and that hence the topological degree of the mapping described by $P_1, \cdots, P_m$ is the sum of the signs of the Jacobians of these solutions. Thus computing the topological degree yields a lower bound for the number of almost periodic solutions of the varied system. If $m = 2$, the sign of the topological degree yields information about the stability of the solutions.

The technique used is similar to that previously used [3] to study periodic solutions. The essential difference is that in order to prove the existence of periodic solutions, we need only show that a certain topological degree is nonzero whereas to prove the existence of almost periodic solutions, we must first vary function F. Although the variations in F are "arbitrarily small", they are fairly elaborate. Function F regarded as a function of t must be varied and also regarded as a function of x. In physical language, the external force must be varied and the system itself must be varied.

2. **Malkin's Theorem.** Our results are based on the treatment given by Malkin [4, Chapter IV, especially pp. 285–297]. (Page references are to the AEC translation.) So we describe Malkin's result for (E_μ). First the hypotheses on the n-dimensional system (E_μ) are:

1) constant matrix A has m critical roots where $m > 0$;
2) the components of $f(t)$ are finite trigonometric sums;

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