FIXED POINTS AND COVERING UNDER CONTINUOUS MAPPINGS OF A SPHERICAL SHELL

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The notations of the preceding paper, "Fixed points of mappings into Euclidean n-space" will be followed. We use $| C_i |$ to denote the point set locus of an oriented chain C_i, but omit the bars where no misunderstanding is likely to arise.

Hypotheses (H). *Let S_1^{n-1} and S_2^{n-1} be concentric $(n-1)$- spheres in Euclidean n-space R^n, with S_1^{n-1} the larger. Let θ be a homeomorphism from S_1^{n-1} and its interior into R^n, and let K^n be the image under θ of the closed region bounded by S_1^{n-1} and S_2^{n-1}. We assume K^n to be a complex of cells, with all n-cells oriented positively, and let σ_1^{n-1} and σ_2^{n-1} be the oriented $(n-1)$-cycles of the boundary, where $\sigma_j^{n-1} = \theta(S_j^{n-1})$, $j = 1, 2$.*

Following the notation of the preceding paper, by f we denote a continuous mapping of K^n into R^n. In Theorem 1 below we give sufficient conditions under which at least one point of K^n must be fixed under f, and in Theorem 2, sufficient conditions under which either f has a fixed point or $f(K^n)$ covers the entire interior of σ_2^{n-1}. The proofs will use the following lemmas.

Lemma 1. *If f has no fixed point, the turning index of $(\sigma_1^{n-1} + \sigma_2^{n-1})$ is zero.* This is an easy corollary of Lemma 5 of the preceding paper.

Lemma 2. *If q is a point of R^n and $q \notin f(K^n)$, then the index of q relative to $f(\sigma_1^{n-1} + \sigma_2^{n-1})$ is zero.*

Proof. If we take a sufficiently fine subdivision of K^n, the $(n-1)$-cycle on the direction sphere D^{n-1} resulting from f applied to the boundary of each n-cell of the subdivision will be an $(n-1)$-cycle on a small region of D^{n-1}, hence homologous to zero on D^{n-1}. As the index of q relative to $f(\sigma_1^{n-1} + \sigma_2^{n-1})$ is the sum of the indices of q relative to the images under f of the boundaries of the oriented n-cells of K^n, we infer that the lemma is true.

Lemma 3. *If σ^{n-1} is the boundary of a positively oriented n-cell η^n in R^n, with $| \sigma^{n-1} |$ on K^n, and f has no fixed point on σ^{n-1} and, finally, $f(\sigma^{n-1}) \subset | \eta^n |$, then the turning index of σ^{n-1} under f is $(-1)^n$.*

Proof. Let ϕ be a homeomorphism from a closed solid n-sphere E^n with center 0, to η^n and σ^{n-1}. Let a continuous change in the mapping f as applied to σ^{n-1} be made by having the image points under f move along paths which are the images under ϕ of radii, toward $\phi(0)$. It is clear that the turning index of

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