INTERVAL FUNCTIONS AND THE HELLINGER INTEGRAL

By William D. L. Appling

1. **Introduction.** Suppose m is a real-valued nondecreasing function on the number interval $[a, b]$.

We establish the following result (Theorem 2): If H is a real-nonnegative-valued function of subintervals of $[a, b]$ such that the integral (§2)

$$\int_{[a,b]} [H(I)\, dm]^{\frac{1}{2}}$$

exists, and there is a nonnegative number R such that $H(I) \leq R\, \Delta m$ for each subinterval I of $[a, b]$, then

$$\int_{[a,b]} \left(\int_I [H(J)\, dm]^{\frac{1}{2}}\right)^2 \bigg/ dm = \int_{[a,b]} H(I).$$

We prove (Theorem 4) that if each of H and K is a real-valued bounded function of subintervals of $[a, b]$ such that each of $\int_{[a,b]} H(I)\, dm$ and $\int_{[a,b]} K(I)\, dm$ exists, then

$$\int_{[a,b]} H(I)K(I)\, dm$$

exists.

We show (Theorem 6) that if H is a real-valued bounded function of subintervals of $[a, b]$ such that $\int_{[a,b]} H(I)\, dm$ exists, and for some number S and every subinterval I of $[a, b]$, $0 < S \leq |\, H(I)\, |$, then

$$\int_{[a,b]} [dm]/H(I)$$

exists.

We demonstrate (Theorem 7) that if g is a real-valued function on $[a, b]$ having bounded variation on $[a, b]$ and H is a real-valued bounded function of subintervals of $[a, b]$, then

$$\int_{[a,b]} H(I)\, dg$$

exists if and only if

$$\int_{[a,b]} H(I) \left[\int_I |\, dg\, |\right]$$

exists.

Received October 16, 1961.