IMPLICATIONS OF POINTWISE BOUNDS ON POLYNOMIALS

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In this paper we prove several theorems which originated with the problem of determining upper bounds for the absolute values of a sequence of polynomials in an interval, given that the sequence converges on a subset of the interval having positive measure. As it turns out, the solution to this problem appears as a consequence of a considerably more general result.

Lemma 1. *Let $I = [a, b]$ be a compact interval in the real line, and let P be a polynomial of degree n with complex coefficients, not all zero. Suppose A is a closed subset of I with $m(A) = k(b - a)$, where m denotes Lebesgue measure and $0 < k \leq 1$. Define*

$$R_A(P) = \frac{\max_{x \in I} | P(x) |}{\max_{x \in A} | P(x) |}.$$

Then

$$\begin{aligned} R_A(P) &\leq R_k(n) \\ &= \tfrac{1}{2}[(2k^{-1} - 1 + 2k^{-1}\sqrt{(1 - k)})^n + (2k^{-1} - 1 - 2k^{-1}\sqrt{(1 - k)})^n]. \end{aligned}$$

If A is a subinterval of I with a common endpoint, this bound is the best possible.

Proof. We can clearly assume $n > 0$, and that P has lead coefficient 1. Now let x_0 be a point in I. To prove the lemma, it suffices to show that in general,

$$\frac{| P(x_0) |}{\max_{x \in A} | P(x) |} \leq R_k(n),$$

and in doing this, we may first of all assume that $x_0 = b$; for in any event, either $m(A \cap [a, x_0]) \geq k(x_0 - a)$ or $m(A \cap [x_0 , b]) \geq k(b - x_0)$, and in the first case, we simply consider the interval $[a, x_0]$, while in the second, we substitute $(-1)^n P(a + b - x)$ for $P(x)$ and consider the interval $[a, a + b - x_0]$.

We may further assume that all the roots of P lie in $[a, b]$, for if not, let $a_1 , \cdots , a_n$ be the roots of P (with proper multiplicities), and let $c_1 , \cdots , c_n$ be real numbers such that $c_i \leq b$, and $| b - a_i | = b - c_i$ $(i = 1, \cdots , n)$. Let $Q(x) = \prod_{i=1}^{n} (x - c_i)$. Then $Q(b) = | P(b) |$, while $| Q(x) | \leq | P(x) |$ for $x \in [a, b]$. I.e., $R_A'(P) \leq R_A'(Q)$, where we define $R_A'(S)$ to be $R_A(S)$ with the numerator of the latter replaced by $| S(b) |$. Now if any of the c_i are less than a, replace them by a in the expression for Q, and call the resulting polynomial Q^*.

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