A NEW CONVOLUTION FORMULA AND SOME NEW ORTHOGONAL RELATIONS FOR INVERSION OF SERIES

By H. W. Gould

1. **Introduction.** One of the most well-known orthogonal series relations is

$$\sum_{k=j}^{n} (-1)^{n-k}\binom{n}{k}\binom{k}{j} = \binom{0}{n-j} = \begin{cases} 0, & j \neq n \\ 1, & j = n \end{cases}. \tag{1.1}$$

This immediately yields the well-known inverse series relations

$$F(n) = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} f(k), \tag{1.2}$$

and

$$f(n) = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} F(k). \tag{1.3}$$

There are, of course, many such relations known. The purpose of this paper is to present some new relations which include many of the well-known relations as special cases.

We shall need two preliminary results. Firstly, if $P(x)$ is a polynomial in x of degree less than n, then

$$\sum_{k=0}^{n} (-1)^{k}\binom{n}{k} P(a + bk) = 0, \qquad n \geq 1. \tag{1.4}$$

This is merely the familiar fact that the n-th difference of such a polynomial must be zero.

Secondly, we shall make use of a set of inverse series relations recently proved by the author [3]. These relations are as follows:

$$\binom{a + bn}{n} f(n) = \sum_{k=0}^{n} (-1)^{k} \frac{a + bk - k}{a + bn - k}\binom{a + bn - k}{n - k} F(k), \tag{1.5}$$

where a and b are real numbers and

$$F(n) = \sum_{k=0}^{n} (-1)^{k}\binom{n}{k}\binom{a + bk}{n} f(k), \qquad f(0) = 1. \tag{1.6}$$

2. **The new relations.** We make the definitions

$$R_n = \sum_{k=0}^{n} (-1)^{k} A_k(a, b) A_{n-k}(a + bk - k, 0), \tag{2.1}$$

Received August 7, 1961. Research supported by National Science Foundation Grant G-14095.