SUBINVARIANT MEASURES FOR MARKOFF OPERATORS

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1. **Summary.** Let $(X, \mathcal{X}, \lambda)$ be a σ-finite measure space, and P a nonnegative contraction on $\mathcal{L}_\infty(\lambda)$, such that $f_n \downarrow 0$ implies $Pf_n \downarrow 0$, λ-almost everywhere. Such operators are discussed by E. Hopf in [4], and we shall call them Markoff operators on $\mathcal{L}_\infty(\lambda)$. We shall investigate the existence of P-invariant and P-subinvariant measures for such a P.

DEFINITION. A measure $\mu \prec \lambda$ is called *(sub)-invariant* or *P-(sub)-invariant* if $\int Pf\, d\mu\ (\leq) \int f\, d\mu$ for all $f \in \mathcal{L}_\infty^+(\lambda)$. The qualifying term "on S" will be used if such an equality (or inequality) holds only for f with support in the set S, where S is a fixed set of $\mathcal{X}$.

Hopf has shown how to split X into a "conservative" part C and a "dissipative" part D. In §2 we shall split D further into A and B. A will be (roughly speaking) the points of D from which one *cannot* get to C, and $B = D - A$. If $\mathcal{L}_\infty(\lambda)$ is split into $\mathcal{E}_A + \mathcal{E}_B + \mathcal{E}_C$, where $\mathcal{E}_i = \{f \mid f \text{ has its support in } i\}$, then P has reducibility properties which can be summarized by writing it as a matrix:

	$\mathcal{E}_A$	$\mathcal{E}_B$	$\mathcal{E}_C$
$\mathcal{E}_A$		0	0
$\mathcal{E}_B$			
$\mathcal{E}_C$	0	0	

In §3 it is shown (Corollary a) that a P-subinvariant measure must be P-invariant on C (this is a generalization of a theorem of E. Hopf in [4] and E. Nelson in [5]). A corollary of this is that any P-subinvariant measure on $\mathcal{X}$ assigns measure 0 to B. In view of the reducibility of P, this shows that all we need consider are the two extreme cases of the purely conservative operator (where $C = X$) and the purely dissipative operator (where $D = X$).

For a *dissipative* operator P on $\mathcal{L}_\infty(\lambda)$, it is easy to see that there exists at least one subinvariant measure equivalent to λ (§4), although perhaps there are no invariant ones (see the example in [5]).

In §5, the "process-on-R" is studied. This device, essentially due to Halmos, was used by Harris in [3] to go from an invariant measure on a subset to a measure on the whole space. Some properties of this correspondence are needed for the subsequent sections.

For a *conservative* operator, there may well be *no* subinvariant measure.

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