CONTINUITY OF HOMOMORPHISMS

By R. M. Dudley

In this paper I will establish what seem to be fairly general hypotheses on topological groups G and H guaranteeing that every homomorphism of G into H be continuous. In nearly all cases the condition on H, which is given the discrete topology, is that one can define on it a "norm", by which I mean a nonnegative integer-valued function $p(\ )$ on H with the following properties:

1) $p(hh') \leq p(h) + p(h')$ for all $h, h' \in H$.
2) $p(1) = 0$, where 1 is the unit element of H.
3) $p(h^{-1}) = p(h)$ for all $h \in H$.
4) $p(h^n) \geq \max(n, p(h))$ for any positive integer n and any $h \neq 1$.

A *normed group* will be a pair (G, p) where G is a non-trivial group and p is a norm on G. Here are two facts to help delineate the class of normed groups:

LEMMA 1. *A normed group has no element $h \neq 1$ which is of the form $h_i^{n_i}$ for arbitrarily large n_i, and (consequently) no element of finite order other than* 1.

Proof. The assertion follows directly from 4). The converse is false, as will be seen later.

LEMMA 2. *The collection of normed groups contains the additive group Z of integers and is closed under the formation of (restricted) free products and direct sums (of any cardinality).*

Proof. For Z it suffices to set $p(n) = |n|$. Let $\{(G_\alpha, p_\alpha)\}$ be any collection of normed groups. For an element $\{g_\alpha\}$ of the direct sum $\bigoplus_\alpha G_\alpha$, we set $p(\{g_\alpha\}) = \sum_\alpha p_\alpha(g_\alpha)$, where by assumption there are only finitely many terms in the sum; the verification of properties 1)–4) is straightforward. If g is an element of the free product of the G_α other than 1, it is uniquely of the form $g = g_{\alpha_1} \cdots g_{\alpha_r}$ where $g_{\alpha_i} \in G_{\alpha_i}$ for each i, $\alpha_i \neq \alpha_{i+1}$, $r \geq 1$, and $g_{\alpha_i} \neq 1_{\alpha_i}$ for each i, where 1_{α_i} is the identity in G_{α_i}. We set $p(g) = \sum_i p_{\alpha_i}(g_{\alpha_i})$ and $p(1) = 0$. All conditions in the definition of norm are clearly satisfied except, perhaps, for 4). To prove 4) for a fixed element g, we may assume that there are only finitely many G_α, and hence it suffices to consider a free product of two groups (G, p_1) and (H, p_2). If $\gamma = g_1 h_1 \cdots g_r h_r$ is an element of $G \circ H$ (where $\circ$ denotes free product), we have $p(\gamma) = \sum_i p_1(g_i) + p_2(h_i)$; here no g_i or h_i except perhaps g_1 or h_r is an identity. 4) clearly holds for γ unless g_1 or h_r is an identity, say h_r. Take the smallest k such that $g_k \neq g_{r+1-k}^{-1}$ or $g_k = g_{r+1-k}^{-1}$ and $h_k \neq h_{r-k}^{-1}$, say the former; then

$$\gamma = g_1 h_1 \cdots h_{k-1}(g_k \cdots g_{r+1-k})h_{k-1}^{-1} \cdots g_1^{-1},$$

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