ON THE CHARACTERISTIC ROOTS OF POWER-POSITIVE MATRICES

By Alfred Brauer

A square matrix of order n is called positive if all its elements are positive. Non-negative matrices and negative matrices are defined correspondingly. If B is a negative matrix with the characteristic roots $\omega_1, \omega_2, \cdots, \omega_n$, then $A = -B$ is a positive matrix with the roots $-\omega_1, -\omega_2, \cdots, -\omega_n$.

The properties of the characteristic roots of positive and non-negative matrices were studied by O. Perron [11] and G. Frobenius [8], [9]. In particular, they proved that a positive matrix has a positive characteristic root which is simple and greater than the absolute value of the other roots. This greatest root is greater than the greatest main diagonal element, it is greater than or equal to the smallest row-sum and less than or equal to the greatest row sum. The coordinates of a characteristic vector belonging to the greatest positive root can be chosen as positive numbers.

Other proofs for these results or for some of them were given by P. Alexandroff and H. Hopf [1], H. Wielandt [14], G. Debreu and I. N. Herstein [7], A. S. Householder [10], J. L. Ullman [13], H. Samelson [12], and myself [5]. In my paper not only the existence of a greatest positive root is proved, but the proof gives a method to compute this greatest root and coordinates of a characteristic vector belonging to it as exactly as needed. Later I improved this method somewhat [6].

Since the characteristic roots of a matrix are changed continuously if the elements of the matrix are changed continuously, it can be expected that certain matrices which have mostly positive elements and a few negative elements of relatively small absolute values will have similar properties as positive matrices.

On the other hand, there exist matrices with a greatest positive root ω which is greater than the absolute value of all the roots, for which the coordinates of a characteristic vector belonging to ω cannot all be positive and for which ω is greater than the greatest row-sum. Consider for instance the matrix

$$A = \begin{bmatrix} 10 & -2 & -1 \\ 1 & 1 & 4 \\ -3 & 1 & 1 \end{bmatrix}. \tag{1}$$

The characteristic roots of this matrix (see [2]) lie in the interior of the circles

$$|z - 10| \leq 3; \qquad |z - 1| \leq 5; \qquad |z - 1| \leq 4.$$

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