# THE INVERSE OF CERTAIN FORMULAS INVOLVING BESSEL FUNCTIONS

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1. The well-known formula [3; 148]

$$J_\mu(az)J_\nu(bz) = \frac{(\frac{1}{2}az)^\mu(\frac{1}{2}bz)^\nu}{\Gamma(\nu+1)} \cdot \sum_{m=0}^\infty (-1)^m \frac{(\frac{1}{2}az)^{2m}}{m!\,\Gamma(\mu+m+1)} F\left(-m, -\mu-m; \nu+1; \frac{b^2}{a^2}\right) \tag{1.1}$$

admits an inverse of the form

$$(\tfrac{1}{2}z)^{\mu+\nu} = \sum_{n=0}^\infty C_n J_{\mu+n}(az)J_{\nu+n}(bz). \tag{1.2}$$

We shall determine the $C_n$ explicitly.

Since

$$P_n^{(\mu,\nu)}(x) = \frac{(1+\nu)_n}{n!}\left(\frac{x-1}{2}\right)^n F\left(-n, -\mu-n; \nu+1; \frac{x+1}{x-1}\right),$$

where $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial of degree $n$, (1.1) can be written in the form

$$J_\mu(az)J_\nu(bz) = (\tfrac{1}{2}az)^\mu(\tfrac{1}{2}bz)^\nu \cdot \sum_{m=0}^\infty (-1)^m \frac{(\frac{1}{4}(b^2-a^2)z^2)^m}{\Gamma(\mu+m+1)\Gamma(\nu+m+1)} \cdot P_m^{(\mu,\nu)}\left(\frac{b^2+a^2}{b^2-a^2}\right). \tag{1.3}$$

Then

$$\begin{aligned}
\sum_{r=0}^\infty (-1)^r A_r (ab)^{-r}(b^2-a^2)^r J_{\mu+r}(az)J_{\nu+r}(bz)
&= (\tfrac{1}{2}az)^\mu(\tfrac{1}{2}bz)^\nu \sum_{r=0}^\infty (-1)^r (b^2-a^2)^r(\tfrac{1}{2}z)^{2r} A_r \\
&\quad \cdot \sum_{m=0}^\infty (-1)^m \frac{(b^2-a^2)^m(\frac{1}{2}z)^{2m}}{\Gamma(\mu+r+m+1)\Gamma(\nu+r+m+1)} \\
&\quad \cdot P_m^{(\mu+r,\nu+r)}\left(\frac{b^2+a^2}{b^2-a^2}\right) \\
&= (\tfrac{1}{2}az)^\mu(\tfrac{1}{2}bz)^\nu \sum_{n=0}^\infty (-1)^n \frac{(b^2-a^2)^n(\frac{1}{2}z)^{2n}}{\Gamma(\mu+n+1)\Gamma(\nu+n+1)} \\
&\quad \cdot \sum_{r=0}^n A_r P_{n-r}^{(\mu+r,\nu+r)}\left(\frac{b^2+a^2}{b^2-a^2}\right).
\end{aligned}$$

Received December 2, 1960.