REPEATED INTEGRALS OF THE SQUARE-WAVE FUNCTIONS AND RELATED SETS OF ORTHOGONAL FUNCTIONS.

By John W. Cell and Walter J. Harrington

1. **Introduction.** We define

$$S(1; x) = 1, \qquad 0 < x < 1,$$
$$= -1, \qquad 1 < x < 2,$$
$$= 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1,$$

and

$$S(1; x + 2) = S(1; x).$$

In the article we present interesting and useful properties of repeated integrals of this function and of the related function

$$C(1; x) = S(1; x + \tfrac{1}{2}).$$

The sequences of functions, $\{S(1; nx)\}$ and $\{C(1; nx)\}$, have been considered (in a slightly different notation) in a previous article [3] which also contains a more extensive bibliography. We also construct sets of functions which are expressible linearly in terms of corresponding integrals of these S and C functions and which are orthogonal and complete.

It is easily shown that the Fourier series for these two functions are given by

$$S(1; x) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k} \sin k\pi x, \qquad C(1; x) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{\cos k\pi x \sin (k\pi/2)}{k}.$$

We define for $m > 1$,

$$C(m; x) = -g_m \int_{-\frac{1}{2}}^{x} S(m - 1; u)\, du, \qquad S(m; x) = g_m \int_0^x C(m - 1, u)\, du,$$

or by the related Fourier series:

$$C(2n; x) = c_{2n} \sum_{k \text{ odd}} \frac{\cos k\pi x}{k^{2n}}, \qquad S(2n; x) = c_{2n} \sum_{k \text{ odd}} \frac{\sin k\pi x \sin k\pi/2}{k^{2n}},$$

$$C(2n + 1; x) = c_{2n+1} \sum_{k \text{ odd}} \frac{\sin (k\pi/2) \cos k\pi x}{k^{2n+1}}, \quad S(2n + 1; x) = c_{2n+1} \sum_{k \text{ odd}} \frac{\sin k\pi x}{k^{2n+1}}.$$

Evidently $C(m; x)$ and $S(m; x)$ are each periodic of period $x = 2$, $C(m; x)$ is an even function, and $S(m; x)$ is an odd function.

We choose c_m (or g_m) so that $C(m; x)$ and $S(m; x)$ each have an amplitude

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