FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—XII: COMPLEX OPERATORS AND MIN-MAX VARIATION

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1. Introduction. In [1], we applied the functional equation approach of dynamic programming [2] to the study of the variational problem associated with the Sturm-Liouville equation

$$(p(x)u')' + (r(x) + \lambda g(x))u = 0, \qquad u(a) = u(1) = 0, \tag{1}$$

where p, g, and r are real. In this way, we were able to obtain the dependence of the Green's function upon the end-point a. From this result, we obtained the dependence of the characteristic values and characteristic functions upon a. Corresponding results, the Hadamard variation formulas, were presented for partial differential equations in [3].

In order to apply the functional equation technique, we were required to assume that p, q, and r were real. In [4], we indicated briefly how min-max techniques could be used to study the corresponding problems for complex functions. In this paper, we wish to present the full argument for a particular class of equations of the foregoing form.

2. Boundary value problem. Let us consider the following boundary value problem:

$$\begin{aligned} U''(x) + q(x)U(x) &= v(x), \qquad && (a < x < b), \\ U(a) &= 0, && U(b) = 0, \end{aligned} \tag{1}$$

where $q(x)$ and $v(x)$ are *complex-valued* functions continuous on the closed interval $[a, b]$. If the Sturm-Liouville problem obtained by replacing q by λq does not have $\lambda = 1$ as a characteristic value, then the boundary value problem has a unique solution given by the equation

$$U(x) = \int_a^b K(x, y, a)v(y)\, dy \tag{2}$$

where $K(x, y, a)$ is the Green's function of the problem. We shall regard b as fixed and study the dependence of the Green's function upon a.

In order to use an approach suggested by dynamic programming, we shall find it convenient to study the system

$$\begin{aligned} u''(x) + q(x)u(x) &= v(x), \qquad && (a < x < b), \\ u(a) &= c, && u(b) = 0. \end{aligned} \tag{3}$$

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