REAL CHARACTERS OF CERTAIN SEMI-GROUPS WITH APPLICATIONS

By A. Devinatz and A. E. Nussbaum

1. **Introduction.** In this paper we mean by an *open semi-group* an open subset $\mathfrak{S}$ of a topological group $\mathfrak{G}$ (which is not assumed to be locally compact or Abelian) which is a semi-group with respect to multiplication defined in $\mathfrak{G}$. That is, $xy \in \mathfrak{S}$ if $x \in \mathfrak{S}$ and $y \in \mathfrak{S}$. In §2 we show that every homomorphism of an open semi-group into the multiplicative semi-group of real numbers which is positive and bounded in an open set of $\mathfrak{S}$ is continuous, and various other related theorems. In §3 we apply the results of §2 to a semi-group $\{T_x\,;\, x \in \mathfrak{S}\}$ of bounded self-adjoint operators T_x on a Hilbert space, and obtain an integral representation of such a semi-group of operators. In §4 we prove that if the open semi-group $\mathfrak{S}$ has the identity e of $\mathfrak{G}$ as a contact point, $\mathfrak{G}$ is locally compact (with an additional restriction) and a semi-group $\{T_x\,;\, x \in \mathfrak{S}\}$ of bounded self-adjoint operators is weakly measurable with respect to the measure induced on $\mathfrak{S}$ by the Haar measure of $\mathfrak{G}$, then $\{T_x\}$ is strongly continuous. In §5 we apply the results of §3 to completely monotonic functions on an open semi-group and obtain an integral representation of such functions. In so doing we greatly generalize, for open semi-groups, the Hausdorff-Bernstein-Widder theorem proved earlier by one of the authors [12].

In §6 we show that the results of §5 can be used to obtain a similar theorem for functions whose values lie in a partially ordered vector space. In so doing we generalize theorems due to S. Bochner [2] and E. J. McShane [10] (see also Choquet [3]).

2. Real characters of an open semi-group.

Definition 1. *A real character of an open semi-group $\mathfrak{S}$ is a continuous homomorphism χ of $\mathfrak{S}$ into the multiplicative semi-group R of real numbers.*

Theorem 1. *If χ is a homomorphism of an open semi-group $\mathfrak{S}$ into the multiplicative semi-group of real numbers and if χ is positive and bounded on an open set $\mathfrak{O} \subseteq \mathfrak{S}$, then χ is continuous on $\mathfrak{S}$, i.e., χ is a real character of $\mathfrak{S}$.*

Proof. Let $\mathfrak{G}$ be the topological group of which $\mathfrak{S}$ is an open subspace and e the identity element of $\mathfrak{G}$. Let $\mathfrak{P}$ be the set of all elements x in $\mathfrak{S}$ such that $\chi(x) > 0$. Let $f(x) = \log \chi(x)$ for $x \in \mathfrak{P}$, then $f(xy) = f(x) + f(y)$ for all x, y in $\mathfrak{P}$.

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