CONTRACTIBILITY IN SPACES OF HOMEOMORPHISMS

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1. **Introduction and results.** Suppose X is compact, metric, and of finite dimension n. For each positive integer k let $\mathcal{H}(X, I^k)$ be the space of all homeomorphisms of X into I^k, the k-dimensional interval, and $\mathcal{C}(X, I^k)$ the space of all mappings (continuous functions) of X into I^k, with the usual metric topology. ($\rho(f, g) = \max d(f(x), g(x))$ for $x \in X$, where d is the metric in I^k and ρ is the metric in $\mathcal{C}(X, I^k)$.) We also consider $\mathcal{C}(X, I^\omega)$ and $\mathcal{H}(X, I^\omega)$, where the Hilbert cube replaces I^k, and in this case X is not required to be finite dimensional. In [2] it was stated that $\mathcal{H}(X, I^k)$ is arc-wise connected, and locally arc-wise connected, if $k \geq 2n + 2$. In [3] a result was stated which is more general in two directions, as indicated in the following theorem.

THEOREM 1. *Suppose S^r is a topological r-sphere in $\mathcal{C}(X, I^k)$ and $k \geq 2n + 2 + r$. Then S^r is contractible in $S^r \cup \mathcal{H}(X, I^k)$. That is to say, there exists a mapping $F: S^r \times I \to \mathcal{C}(X, I^k)$ such that*

(i) *$F(f, 0) = f$ for all $f \in S^r$,*

(ii) *$F(f, t) \in \mathcal{H}(X, I^k)$ if $0 < t \leq 1$,*

(iii) *$F(f, 1) = g$ where g is fixed (independent of f).*

Other results are as follows.

THEOREM 2. *Suppose T is an $(r + 1)$-dimensional polytope, $\dim X = n$, and $k \geq 2n + 2 + r$. Then the space A of all mappings of T into $\mathcal{C}(X, I^k)$ contains a dense G_δ set B of mappings of T into $\mathcal{H}(X, I^k)$.*

THEOREM 3. *Suppose X and K are compact metric spaces, $\dim X = n$, $\dim K = r$, and $k \geq 2n + 2 + r$. Let α_0 and α_1 be mappings of K into $\mathcal{C}(X, I^k)$. Then there exists a homotopy $f: K \times I \to \mathcal{C}(X, I^k)$ such that* (1) *$f(w, 0) = \alpha_0(w)$, $f(w, 1) = \alpha_1(w)$, $(w \in K)$, and* (2) *for all t $(0 < t < 1)$, $f(w, t) \in \mathcal{H}(X, I^k)$. In fact, if A is the space of all homotopies of α_0 to α_1 in $\mathcal{C}(X, I^k)$, then A contains a dense G_δ subset B such that every f in B satisfies* (1) *and* (2) *above.*

THEOREM 4. *Without any restrictions on the dimensions of X and K, if in Theorem 3 I^k is replaced by I^ω, the resulting statement is true.*

Theorem 1 is a special case of Theorem 3, as may be seen by taking $K = S^r$, α_0 the identity mapping and α_1 a constant mapping of K into $\mathcal{C}(X, I^k)$. Theorem 2 is considered primarily as a lemma used in the proof of Theorem 3.

Most of this paper is devoted to a proof of Theorem 3. In contrast, Theorem 4 is easy to prove. In fact, if $P = K \times X$, then Theorem 4 follows quickly from

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