A CONTINUITY PROPERTY FOR ADJOINTS OF CLOSED OPERATORS IN BANACH SPACES, AND ITS APPLICATION TO ELLIPTIC BOUNDARY VALUE PROBLEMS

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In some recent papers [6], [7], [9], the writer has studied the solvability of of a general family of elliptic boundary-value problems in L^p for $1 < p < +\infty$. The principal tool in this study has been a corresponding family of *a priori* estimates of the form

$$\sum_{|\alpha| \leq 2m} \| D^\alpha u \|_{L^p} \leq C\{\| u \|_{L^p} + \| Au \|_{L^p}\},$$

holding for all functions u satisfying some given boundary conditions $\{B_j u = 0\}$ on the boundary of a given domain G, where A is an elliptic operator of order $2m$ on G, and the boundary-value problem (A, B_j) is regular in a sense which we will specify below. (D^α denotes as usual $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where α is an n-tuple $(\alpha_1, \cdots, \alpha_n)$ of non-negative integers, $D_j = i^{-1}\,\partial/\partial x_j$). Estimates of this type have been obtained for the Dirichlet problem by Koshelev [15], and independently for the general regular problems by the writer [6], [8] and Agmon, Douglis, and Nirenberg [2]. (For $p = 2$, they have also been obtained by Schechter [18] and Peetre [17]. The existence theory in L^2 has been studied by Schechter [19], and in L^p by Agmon [1], the latter using somewhat different methods than the writer in the papers listed above and requiring more stringent assumptions upon the differentiability of the coefficients. L^p estimates for general regular problems were also obtained by Slobodetski, Doklady Akademii Nauk SSSR, vol. 123 (1958) pp. 616–619.)

It is our purpose in the present paper to deal with a problem in functional analysis which arises in the discussion of the existence theory for $p \neq 2$. The procedure which the writer has used to obtain solutions for elliptic equations $Au = f$ where A and its formal adjoint A' have merely continuous coefficients, for example, has been to approximate A by operators A_k with very regular coefficients and to take the limit of the corresponding solutions u_k of $A_k u_k = f$. This procedure is far from straightforward since in general A does not have a bounded inverse under the given boundary conditions and, as far as one knows at this point, for many problems there may be no points in the resolvent set of A in any L^p. We have resolved these difficulties in [9] completely for $p = 2$, and with an additional regularity condition on the coefficients of A if $p \neq 2$. (This regularity condition demands essentially that the coefficients of A should lie in C^1 in the neighborhood of each point of the closure of G.)

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