EXTREMA OF FUNCTIONS OF A REAL SYMMETRIC MATRIX IN TERMS OF EIGENVALUES

By K. A. Bush and I. Olkin

1. Introduction. In this paper we shall establish some known inequalities and a number of new results as special cases of rather powerful general theorems concerning bilinear forms of the real symmetric matrix $A = (a_{ij})$. These general theorems can be continued in an endless hierarchy. Basically they rest upon an ingenious construction originally due to Paley [5]. Paley proposed the problem (essentially) of constructing an $n \times n$ orthogonal matrix using only the entries $1/\sqrt{n}$ and $-1/\sqrt{n}$. We shall refer to such matrices as Paley matrices. It is known that such matrices exist if $n \equiv 0(4)$ for $n \leq 100$ except $n = 92$ (as yet not known), [7].

We shall denote the principal theorems by I, II, $\cdots$ and the particular theorems derived from the general results by numbers such as I.1 if the result follows from I.

2. Bounds for a bilinear form. Quadratic forms have long been studied, and a number of interesting inequalities have been obtained. A number of references to this classic approach are contained in [3] in the references. By considering bilinear forms, however, we are able to secure an essential unity of treatment which is both simpler and more productive as a source of theorems.

Suppose we wish to bound x_1Ax_2' subject to the constraints $x_1x_1' = x_2x_2' = 1$, $x_1x_2' = \rho$ with $|\rho| \leq 1$ where x_1 and x_2 are $1 \times n$ vectors. It is convenient to display the constraints in matrix form, and we shall refer to such an array as the vector correlation matrix R with an appropriate subscript. Here

$$R_2 = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 1 & \rho \\ x_2 & \rho & 1 \end{array}.$$

Let P_2 denote the Paley matrix of order 2:

$$P_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then $S_2 = P_2R_2P_2$ is the diagonal matrix $(1 + \rho, 1 - \rho)$. (It is this diagonalization which makes the Paley matrix such an effective tool). Making the transformation,

$$(x_1, x_2) = (y_1, y_2)P_2,$$

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