A CHARACTERIZATION OF INTEGER GROUPS AND REAL GROUPS

By Hsin Chu

1. Introduction. Let G be a topological group. A subgroup H of G is called *syndetic* if there exists a compact subset K of G such that $G = H \cdot K$. A subgroup, C_x, of G is called a *cyclic subgroup* if C_x is a cyclic group generated by an element x of G.

It will be interesting to consider the following problem in Topological Dynamics.

"What is the structure of a locally compact group G which contains a syndetic cyclic group but contains no compact subgroup other than identity?"

The author will show in this paper that a locally compact group of this type is nothing but either (a) real group R, the additive group of all real numbers with the usual topology or (b) integer group I, the additive group of all integers with the discrete topology.

In this paper, a topological group will be denoted by G, and e (0 in the Abelian case) will denote either the identity element of G or the trivial subgroup consisting of the identity only. Let $x \in G$. The cyclic group generated by x will be denoted by C_x. The additive group of all integers with the discrete topology will be denoted by I, and the additive group of all real numbers with the usual topology will be denoted by R. A *topological isomorphism* between two topological groups, G_1 and G_2, is simultaneously an algebraic isomorphism and a homeomorphism and we shall denote its existence by $G_1 \cong G_2$. If G_1 and G_2 are two topological groups, their *direct product* $G_1 \times G_2$ is the algebraic direct product group endowed with the Cartesian product topology. Let X_1 and X_2 be two topological spaces; if X_1 is homeomorphic with X_2, we will denote them by $X_1 \approx X_2$. Let A and B be closed subgroups of a topological group G. The *double coset space* $(G/A)/B = \{A\ x\ B \mid x \in G\}$ of G is a topological space, by considering $\{A\ U_x\ B \mid x \in G_2 U_x$ is an open neighborhood of x in $G\}$ as the family of all open sets. All other terminologies and notations which are not mentioned here are taken from [1], [4], [6], [8], and [9].

2. Fundamental properties.

Definition 1. A topological group G is said to have *condition* (A) if G contains a syndetic, non-trivial cyclic subgroup C_x, generated by $x \in G$ but $x \neq e$ and G contains no compact subgroup except identity.

Received June 16, 1960. This paper was a part of my doctoral dissertation at University of Pennsylvania. I wish to express my gratitude for the encouragement and direction of Professor W. H. Gottschalk and wish to acknowledge my debt to Professor M. Goto for his valuable advice, continued assistance and helpful suggestions. I am also indebted to Professors M. Suzuki and H. C. Wang for their useful suggestions.