COEFFICIENT IDENTITIES DERIVED FROM EXPANSIONS OF ELEMENTARY SYMMETRIC FUNCTION PRODUCTS IN TERMS OF POWER SUMS

By Jack Levine

1. Introduction. The expansions under consideration are represented by

$$U \equiv a_{p_1}a_{p_2} \cdots a_{p_k} = \sum A\binom{p_1p_2 \cdots p_k}{n_1n_2 \cdots n_r} s_1^{n_1}s_2^{n_2} \cdots s_r^{n_r}, \tag{1.1}$$

where the sum is over all partitions $[1^{n_1} 2^{n_2} \cdots r^{n_r}]$ of the weight $w = p_1 + p_2 + \cdots + p_k = n_1 + 2n_2 + \cdots + rn_r$. We assume $p_1 \geq p_2 \geq \cdots \geq p_k$. The elementary symmetric functions (or unitary functions) a_p are defined as usual from the identity

$$(x - r_1) \cdots (x - r_n) \equiv x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n , \tag{1.2}$$

and the power sums (or one-part functions), by $s_p \equiv \sum r_i^p$. The expansion (1.1) is commonly known as the US expansion in the David-Kendall notation [1].

The purpose of this paper is to obtain relations between the coefficients A of (1.1). The case $k = 1$ is, of course, well known as the inverse of Waring's formula. In this case, MacMahon [5; 6], (1.1) reduces to

$$a_p = \sum \frac{(-1)^{N+p}}{n_1! \cdots n_r!\, 1^{n_1}2^{n_2} \cdots r^{n_r}} s_1^{n_1}s_2^{n_2} \cdots s_r^{n_r}, \tag{1.3}$$

$N = n_1 + \cdots + n_r$, and we have the simple explicit formula for the individual coefficients,

$$A\binom{p}{n_1 \cdots n_r} = \frac{(-1)^{N+p}}{n_1! \cdots n_r!\, 1^{n_1}2^{n_2} \cdots r^{n_r}}. \tag{1.4}$$

It is known that the coefficients of (1.4) are such that their sum is zero, ($p > 1$), and the sum of their absolute values is 1, i.e.,

$$\sum_{n_i} A\binom{p}{n_1 \cdots n_r} = 0, \qquad (p > 1), \tag{1.5}$$

$$\sum_{n_i} \left| A\binom{p}{n_1 \cdots n_r} \right| = 1. \tag{1.6}$$

Proofs may be found in E. Roe [11], Dwyer [2], Ostrowski [8]. It follows from (1.5), (1.6) that the general coefficients of (1.1) satisfy similar identities,

$$\sum_{n_i} A\binom{p_1p_2 \cdots p_k}{n_1n_2 \cdots n_r} = 0, \qquad (U \neq a_1^w = s_1^w), \tag{1.7}$$

Received May 24, 1960.