A GENERALIZATION OF THE TRACE CONCEPT

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1. Introduction. If two operators, T and V, both belong to the trace class, then the sum $T + V$ also does. Let μ_n be the eigenvalues of T ordered according to magnitude, each listed a number of times equal to its multiplicity. Let λ_n be the eigenvalues of $T + V$ similarly ordered. Then the sum $\sum (\lambda_n - \mu_n)$ will converge to the trace of V. (Unless otherwise indicated, summations are extended from one to infinity.) However, it is possible for this sum to exist in cases where neither T nor V is in the trace class. In some problems this sum serves as a "generalized trace" of V. Of course, this generalized trace could depend on both T and V, and we shall see by examples that it actually does so.

Gelfand and Levitan [2] considered the special case where $T = -d^2/dx^2$ with boundary conditions $y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0$, and V is the multiplier operator $q(x)\cdot$. The function $q(x)$ is assumed to be differentiable and to have an average value of 0 over the interval $[0, \pi]$. Using differential and integral equation theory they obtained the result

$$\sum (\lambda_n - \mu_n) = \tfrac{1}{4}[q(0) + q(\pi)]$$

for $h = H = 0$. They state that

$$\sum (\lambda_n - \mu_n) = \tfrac{1}{4}[q(0) + q(\pi)] + hH$$

in general. An obvious counterexample is given by $q(x) \equiv 0$, $hH \neq 0$.

Motivated by the Gelfand-Levitan result we were led to the theorem that $\sum (\lambda_n - \mu_n)$ is equal to the sum of the first order perturbation terms for a wide class of operators. In more detail, if we let $\lambda_n(s)$ be the eigenvalues of $T + sV$, then $\lambda_n(s)$ can be expressed as a power series $\lambda_n(s) = \mu_n + s(V\phi_n, \phi_n) + \cdots$ where ϕ_n is a normalized eigenvector of T corresponding to μ_n. Then

$$\sum (\lambda_n - \mu_n) = \sum (V\phi_n, \phi_n).$$

This result is expressed in Theorem 2 below.

In the course of the proof we obtained another expression for $\sum (\lambda_n - \mu_n)$; namely

$$\sum (\lambda_n - \mu_n) = \lim_{\rho \to \infty} S\{T_\rho^{-1} V T_\rho^{-1}\}$$

where $T_\rho = T + \rho I$ and S denotes the trace function. In all examples considered so far this limit is easier to evaluate than $\sum (V\phi_n, \phi_n)$. For this reason, this characterization of $\sum (\lambda_n - \mu_n)$ is emphasized by stating it separately as Theorem 1 below.

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