# PRIMITIVE IDEMPOTENTS IN THE SEMIGROUP OF MEASURES

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In what follows $S$ is always a compact (Hausdorff) topological semi group with zero. In such a space, an idempotent $\mu$ is *primitive* provided the only idempotents in $\mu S \mu$ are $\mu$ and zero. It is the purpose of this note to investigate the existence of such idempotents in a particular semigroup, namely, the convolution semigroup $S = S(G)$ of normalized regular non-negative Borel measures on a compact (Hausdorff) topological group $G$. More precisely, it is shown that "$\mu$ is a non-zero primitive idempotent in such $S$" is equivalent to each of the statements: (1) carrier $\mu$ is a maximal proper closed subgroup of $G$ and (2) either $\mu$ is central and $\mu S \mu \backslash N = \mu G$ or $\mu S \mu \backslash N = \{\mu\}$, where $N$ denotes the set of nilpotent elements. In addition, if $\mu$ is central, each of the above is equivalent to each of (1)$'$ $S \cdot \mu$ contains precisely two idempotents and (2)$'$ $G/H$ is a finite cyclic group of prime order, where $H$ denotes carrier $\mu$. Finally, some pertinent remarks are made concerning two particular types of compact groups $G$ encountered: those containing a maximal proper closed subgroup and those which contain *no* proper closed subgroups save the identity group. The latter is called *group without proper subgroups*, and an essential lemma shows that such groups are necessarily finite cyclic of prime order. It is also shown that if, in the first type, the maximal proper closed subgroup is normal, then it is open, and several examples are given, one of which indicates how one may construct groups having non-zero primitive idempotents. It may be of interest to note that the relation $\mu \leq \nu$ if $\nu \in \mu S \mu$ gives a partial order on the set $E$ of idempotents of $S(G)$ which makes $E$ into an algebraic lattice (sup $(\mu, \nu)$ = Haar measure on the closed group generated by carrier $\mu$ and carrier $\nu$, and inf $(\mu, \nu)$ = Haar measure on carrier $\mu \cap$ carrier $\nu$) in which primitive idempotents are the maximal elements. More than this $(E, \leq)$ has many topological properties; e.g. it is compact, and sup $(\mu, \nu)$ is continuous when $E$ is Abelian.

For further notation and terminology, [1], [3], [6], and [7] are suggested. The author gratefully acknowledges stimulating conversations with J. G. Horne and L. W. Anderson.

Theorem 1. *Let $S = S(G)$ be the semigroup of measures on the compact group $G$; $\mu \in S$ be a non-zero idempotent. Then, these are mutually equivalent*: (a) *$\mu$ is primitive*, (b) *$H$ = carrier $\mu$ is a maximal proper closed subgroup*, (c) *either $\mu$ is central and $\mu S \mu \backslash N = \mu G$ or $\mu S \mu \backslash N = \{\mu\}$. If, in addition $\mu$ is central, then each of* (a) (b), *and* (c) *is equivalent to either* (d) *$S \cdot \mu$ contains exactly two idempotents or* (e) *$G/H$ is without proper subgroups.*

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