# SOME RESULTS IN SPECTRAL SYNTHESIS

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Let $A$ be the Banach Algebra of absolutely convergent Fourier series, $I$ a closed ideal in $A$, and $S$ the set of common zeros of the functions of $I$.

The assertion of Spectral Synthesis is the implication (1) $f \in A, f(S) = 0 \Rightarrow f \in I$. It is now known, [5], that (1) does not hold unrestrictedly.

In this paper we show that this implication holds provided that $S$ and $f$ are sufficiently restricted. Among other things, these results point out a difference between behavior in 1-dimensional and 3-dimensional space, since Schwartz' counterexample [7] also disproves these results in 3 dimensions.

In all that follows it is assumed that $S$ has measure 0. Since $S$ is clearly closed, we may express its complement as the union of the open intervals $(a_n, b_n)$, $n = 1, 2, \cdots$. Without loss of generality we may also assume that $0 \in S$. We also call $\epsilon_n = b_n - a_n$ and it will prove convenient to introduce the class $\Phi$ of functions $\varphi(t)$ defined and continuous for $t \geq 0$ which are non-negative and non-decreasing and vanish at 0.

Our basic tool is the analysis of the bounded linear functionals, $l$, on $A$, which annihilate all the functions in $I$. We write $l \in I^{\perp}$ for such functionals. Roughly speaking, we derive a "formula" for all such from which spectral synthesis follows formally. To make this formal passage legitimate, finally, the restrictions on $S$ and $f$ are placed.

Our results, then, are the following:

Theorem 1. *Let $l \in I^{\perp}$ then there exists a positive $\delta$ and a sequence $\lambda_n$ such that*

$$l(P) = \lambda_0 P(0) + \sum_{n=1}^{\infty} \lambda_n [P(b_n) - P(a_n)]$$

*for all trigonometric polynomials $P(\theta)$ while*

$$\sum e^{\delta|\lambda_n|} \epsilon_n < \infty .$$

Theorem 2. *Suppose $\varphi \in \Phi$ and*

$$\sum \left| \log \frac{\varphi(\epsilon_n)}{\epsilon_n} \right| \varphi(\epsilon_n) < \infty .$$

*If $l \in I^{\perp}$, then there exists a sequence $\lambda_n$ such that*

$$l(f) = \lambda_0 f(0) + \sum_{n=1}^{\infty} \lambda_n [f(b_n) - f(a_n)]$$

*for all $f \in A$ which satisfy $|f(\theta + t) - f(\theta)| \leq \varphi(t)$.*

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