ECKFORD COHEN'S GENERALIZATIONS OF RAMANUJAN'S TRIGONOMETRICAL SUM $C(n,r)$

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1. Introduction. In [5] Srinivasa Ramanujan studied the trigonometrical sum $C(n,r)$, defined by

$$C(n, r) = \sum_{x} \exp(2\pi inx/r) \tag{1.1}$$

where x runs through a reduced residue system (mod r). Eckford Cohen generalized this sum in [1] as

$$C^{(s)}(n, r) = \sum_{x} \exp(2\pi inx/r^s) \tag{1.2}$$

where x runs through an s-th reduced residue system (mod r^s); that is, all the integers in a complete residue system (mod r^s) whose greatest common divisor with r^s has no s-th power factor greater than 1. He gave in [4] a second generalization $C_{(k)}(n, r)$, by defining a reduced residue system (mod k, r) as the system containing all the ordered sets of k integers $(x_1, x_2, \cdots, x_k)$ from a complete residue system (mod r), with the property that the greatest common divisor of $x_1, x_2, \cdots, x_k$ and r is 1. Then $C_k(n, r)$ is defined by

$$C_{(k)}(n, r) = \sum_{(x_1, x_2, \cdots, x_k)} \exp(2\pi in(x_1 + x_2 + \cdots + x_k)/r) \tag{1.3}$$

where the set $(x_1, x_2, \cdots, x_k)$ runs through a reduced residue system (mod k, r).

It can be shown that $C_{(k)}(n, r) = C^{(k)}(n^k, r)$ using two of Cohen's results viz.

$$(1)\quad C^{(k)}(n, r) = \frac{\phi_{(k)}(r)\mu(r/g_1)}{\phi_{(k)}(r/g_1)}, \quad [2;\text{ Theorem 2}];$$

$$(2)\quad C_{(k)}(n, r) = \frac{\phi_{(k)}(r)\mu(r/g_2)}{\phi_{(k)}(r/g_2)}$$

[4; Theorem 1] where g_1^k is the greatest k-th power common divisor of r^k and n, g_2 is the greatest common divisor of r and n, and $\phi_{(k)}(r)$ is Jordan's extension of Euler's ϕ-function; i.e., the number of sets of k integers $(x_1, x_2, \cdots, x_k)$ form a complete residue system (mod r) whose greatest common divisor with n is 1, and μ is Möbius μ-function.

The object of this paper is to give a further generalization $C_{(k)}^{(s)}(n, r)$ of $C(n, r)$ by combining these two generalizations and to show that $C_{(k)}^{(s)}(n, r)$ and Jordan's function $\phi_{(ks)}(r)$ are connected in a simple relation, viz. Theorem 2. We define $C_{(k)}^{(s)}(n, r)$ as

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