A PRIORI ESTIMATES FOR THE SOLUTIONS OF DIFFERENCE APPROXIMATIONS TO PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. In their basic paper [1], Courant, Friedrichs and Lewy first discovered the important fact that more than a formal connection between difference and differential operators is required for the convergence of finite-difference approximations of solutions of partial differential equations. The additional requirement, now called stability, is that the solutions of the difference equation depend continuously, relative to a suitable norm, on the prescribed data. Stability, and its relation to convergence of finite-difference schemes, has since been discussed from various points of view in the literature; see, for instance, Douglas [3], Lax and Richtmyer [9], O'Brien, Hyman and Kaplan [14], Richtmyer [15] and the bibliographies in [9] and [15].

On the basis of these general theories, most of the finite-difference schemes currently in use for the numerical solution of parabolic and hyperbolic equations with constant coefficients have been successfully analyzed for stability by means of Fourier methods. A table of the standard finite-difference schemes for the equation of heat conduction,

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}, \qquad \sigma = \text{constant} > 0, \tag{1.1}$$

can be found in [15].

Most of the finite-difference schemes for (1.1) can be generalized, in a variety of ways, to parabolic equations with non-constant coefficients. But, even though it is possible to predict the conditions under which they are stable, it is in general a non-trivial task to verify their stability rigorously. One method, the maximum principle, has been successful in a number of important cases. It has been shown by Douglas [4], John [7], Laasonen [8], Lees [13] and Rose [16] that the maximum principle for parabolic equations is shared by several finite-difference schemes, and from this they deduced stability relative to the maximum norm. Unfortunately, many of the standard finite-difference schemes for parabolic equations do not possess a maximum principle (Douglas [5]).

Recently, an investigation of the stability properties of several finite-difference schemes for a class of quasi-linear parabolic equations was undertaken by

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