THE DEFICIENCIES OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN ONE

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Introduction. This paper concerns one aspect of Nevanlinna's theory of meromorphic functions.

We assume that the reader is familiar with the fundamental concepts of this theory, and in particular with the most usual of its symbols:

$$\log^+, \qquad M(r, f), \qquad m(r, f), \qquad n(r, a), \qquad N(r, a), \qquad T(r, f), \cdots .$$

The deficiency of the value τ, with respect to the meromorphic function $f(z)$, is, by definition,

$$\delta(\tau, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, \tau)}{T(r, f)}.$$

If no confusion is to be feared, we write $\delta(\tau)$ instead of $\delta(\tau, f)$.

If $f(z)$ is of finite order λ, there seem to be some connections between λ and the possible values of $\delta(\tau)$ as τ runs through the denumerable sequence of deficient values (that is values of τ for which $\delta(\tau) > 0$).

However, very little is known about the exact nature of the relationship.

One striking result in this direction is due to Pfluger [7; 92], who proved:

Theorem A. *If $f(z)$ is entire and of finite order λ and if*

$$\sum_{\tau\neq\infty} \delta(\tau) = 1,$$

then λ is a positive integer and each $\delta(\tau)$ is an integral multiple of $1/\lambda$.

In the same general direction, the following problem appears as one of the simplest:

Let δ_0 and δ_∞ be two real numbers

$$0 \leq \delta_0 \leq 1, \qquad 0 \leq \delta_\infty \leq 1.$$

When is it possible to find a meromorphic function $f(z)$ of order λ, such that

$$\delta_0 = \delta(0, f), \qquad \delta_\infty = \delta(\infty, f)?$$

In this paper, we solve this problem for $\lambda \leq 1$; our solution is contained in

Theorem 1. *Let $f(z)$ be a meromorphic function of order λ, $0 < \lambda < 1$. Put*

$$u = 1 - \delta(0, f), \qquad v = 1 - \delta(\infty, f).$$

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