DIFFERENCE METHODS FOR MIXED BOUNDARY-VALUE PROBLEMS

By W. Gilbert Strang

1. **Introduction.** We shall consider difference approximations to problems governed by linear parabolic or hyperbolic partial differential equations, when data on the solution is prescribed throughout a finite domain of m-space at time $t = 0$ and on its boundary thereafter. More than a formal correspondence of difference and differential operators is required for *convergence* to the solution when an increasingly fine mesh is used for the difference approximation. This fundamental observation was made in 1928 by Courant, Friedrichs, and Lewy [2]. The additional requirement is *stability*—that the errors introduced at each time-step in the difference scheme have a limited rate of growth.

Lax and Richtmyer established in their important paper [6] the equivalence of convergence and stability for initial-value problems. In mixed problems the situation is similar; however, it may well happen that stability is verifiable in one norm (e.g. l_2) and not in another (e.g. the maximum norm). Our approach to this difficulty is an alternative to one along the lines of Sobolev's lemma, as in Weinberger [8]. It turns out that in the latter norm convergence still holds whenever the data is in a certain sense *smooth* enough; calling this property of a method *s-convergence*, we examine its equivalent, *s-stability*.

The structure of our theory is modeled closely on that of Lax and Richtmyer. We describe permissible difference schemes, define s-convergence and s-stability, prove their equivalence in §8, and then investigate means of verifying s-stability. Finally, an example is given to illustrate the methods of the paper, as well as its applicability to variable-coefficient problems on non-rectangular domains.

2. **Domain and norms.** Let the bounded domain D^-, with simply-connected interior D and boundary D', be a union of closed unit cubes in real m-space whose vertices have integral coordinates. Over D^- we lay a succession of meshes of width $1/q$, $q = 2, 3, \cdots$, and for each q we denote the set of mesh-points in D, D^-, and D' by D_q, D_q^- and D_q', respectively. Let the points of D_q receive a fixed numbering, from 1 to r_q. We further select positive constants C and ρ, and define

$$\Delta t_q = Cq^{-\rho}. \tag{1}$$

The p-norm of a vector $v = (v_1, \cdots, v_r)$ is defined as

$$|v|_p = \left(\frac{1}{r}\Sigma |v_i|^p\right)^{1/p} \qquad 1 \leq p < \infty \tag{2}$$

$$|v|_\infty = \max_{1\leq i\leq r} |v_i| \qquad p = \infty \tag{3}$$

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