THE NUMBER OF PERIODIC SOLUTIONS OF NON-AUTONOMOUS SYSTEMS

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1. **Introduction.** The purpose of this note is to obtain, by using topological degree, some new theorems concerning the existence of periodic solutions for two-dimensional non-autonomous systems of differential equations and to get a lower bound for the number of distinct periodic solutions. The systems treated are of the form:

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= X(x, y) + E_1(t) \\ \frac{dy}{dt} &= Y(x, y) + E_2(t) \end{aligned}$$

where $E_1(t)$, $E_2(t)$ are periodic but not necessarily small. (That is, we do not limit ourselves to perturbation problems.)

Topological methods have been used by numerous writers to study ordinary differential equations. We mention only those papers that are directly related to the results to be described here. Levinson [11], [12], Cartwright and Littlewood [4], [5] and Lefschetz [9] have used the fixed point theorem to obtain existence theorems. Bass [2] has used topological degree, and Gomory [8] has used the index of the vector field. The results to be obtained here are an extension of Part I of Gomory's paper. In terms of Gomory's approach, we compute the index of the vector field and show that the absolute value of the index is a lower bound for the number of distinct periodic solutions. Actually we have rephrased some of Gomory's results in terms of topological degree and we state the new results in the same terms partly because the topological degree is more convenient and partly to emphasize the significance of certain of Gomory's results.

In particular, we show that one of Gomory's results may be rephrased in terms of topological degree in this way: if the usual a priori bound assumption on the solutions is replaced by a similar but usually stronger hypothesis (Assumptions 1 and 2 of §2 are a precise formulation of this hypothesis), then the entire standard procedure of introducing a one-parameter family of mappings (called the method of "global perturbations" by Bass [2]) can be avoided. The problem of determining the topological degree is reduced to that of computing the topological degree of the mapping defined by $X(x, y)$ and $Y(x, y)$. (See Lemma 1 of §2.)

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