THE CHARACTERISTIC DIVISORS OF A POLYNOMIAL FUNCTION OF A MATRIX

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1. Introduction. Let $(F)_n$ denote the ring of $n \times n$ matrices over a field F and suppose A to be a matrix of $(F)_n$ with characteristic polynomial $Q(x)$. Further, let $f(x)$ be an arbitrary polynomial in the ring F_x of polynomials over F. The problem of determining the characteristic divisors of the matrix $B = f(A)$ was first solved in the case of the complex field by Kreis [1] in 1906. Later, in 1935, a solution of this problem in the case of arbitrary fields was given by McCoy [3]. Subsequent proofs of McCoy's results for fields F of characteristic 0 have been given by MacDuffee [2], Williamson [6], and Wagner [5]. For the earlier history of this problem the reader is referred to the papers of McCoy and MacDuffee.

In this paper (§3) we give a fresh solution of the above problem, based on the theory of representation spaces and valid for arbitrary fields F. A statement of the main theorem, as well as certain preliminary lemmas, will be found in §2. In the remainder of this section we sketch briefly the theoretical background of the method employed.

The (F_x, F)-representation space corresponding to the homomorphism, $x \rightarrow A$, mapping F_x onto a subring of $(F)_n$, will be denoted by X [4, §110]. The space X, when viewed as an F_x-module, may be decomposed into a direct sum of cyclic F_x-submodules whose annihilator ideals are generated by powers of (monic) irreducibles in F_x [4, §§109, 111]. The submodules are vector spaces relative to F and hence are cyclic (F_x, F)-subspaces of X. The annihilator polynomials are the characteristic divisors of A and their product is the characteristic polynomial $Q(x)$ of A. The polynomial $\rho(x)$ which generates the minimum ideal, that is, the kernel of the homomorphism $x \rightarrow A$, is the minimum polynomial of A.

Now place $z = f(x)$ and denote by Z the (F_z, F)-representation space corresponding to the homomorphism, $z \rightarrow B$. Since this homomorphism may be extended to the homomorphism $x \rightarrow A$, one may suppose X and Z to consist of identical elements. We may also assume that the matrices A and B refer to a common F-basis (u) of X and Z. The problem of determining the characteristic divisors of B is therefore reduced to the problem of effecting a cyclic, prime-power decomposition of Z. There is no loss of generality in supposing that X is cyclic and indecomposable, that is, A is non-derogatory ($Q(x) = \rho(x)$) and $Q(x)$ is the power of an irreducible of F_x.

2. Preliminaries. We introduce the following notation. Place $Q(x) = P^d(x)$ where $P(x)$ is irreducible of degree h in F_x, $dh = n$. Let λ denote a root of $P(x)$

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