ERGODIC THEORY FOR DISCRETE SEMI-MARKOV CHAINS

By Philip M. Anselone

1. Introduction and summary. The idea of a semi-Markov process is due to K. L. Chung. The only papers on the subject as of January 1959 are those of Lévy [4] and Smith [5].

We consider a semi-Markov chain $\{x_t , t \varepsilon T\}$, where T is the set of non-negative integers and the x_t are countably valued. The semi-Markov chain is defined in terms of a Markov chain $\{\phi_k , k \varepsilon T\}$ and an increasing sequence $\{t_k , k \varepsilon T\}$ of integer valued "transition times" by $x_t = \phi_k$ if $t_k \leq t < t_{k+1}$. Further conditions are included in the complete definition given in §2.

We study this process by a method which differs considerably from those employed by Lévy and Smith. An auxiliary random sequence $\{y_t , t \varepsilon T\}$ is defined by $y_t = 0$ if $t = t_k$ and $y_t = t_{k+1} - t$ if $t_k < t < t_{k+1}$ for some $k \varepsilon T$. Under specified conditions, $\{(x_t , y_t), t \varepsilon T\}$ is a Markov chain. Standard theory is used in the derivation of ergodic properties of this chain. Using these results, we derive ergodic theorems for the semi-Markov chain $\{x_t , t \varepsilon T\}$. Final results relate the ergodic behavior of $\{x_t , t \varepsilon T\}$ and $\{\phi_k , k \varepsilon T\}$.

The method just described also applies to continuous parameter semi-Markov processes. However, the situation is complicated by the appearance of discontinuities at $t = t_k , k \varepsilon T$. Feller encountered the same difficulty in [3; 427], in connection with a similar method. We present no treatment of continuous parameter semi-Markov processes in this paper.

All random variables introduced below are defined with respect to a fixed probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Statements involving conditional probabilities or expectations have the implicit qualification, "whenever the symbols are defined." For the pertinent general theory of Markov chains, see Feller [2] or Doob [1].

2. Definition of the semi-Markov chain. Let $\{\phi_k, k \varepsilon T\}$ denote a Markov chain with stationary transition matrix $[p_{im}]$ and values in a countable set I. Thus,

$$\mathbf{P}\{\phi_{k+1} = m \mid \phi_k = i\} = p_{im} . \tag{1}$$

Let $t_k , k \varepsilon T$, denote integer-valued random variables such that

$$\theta_k = t_{k+1} - t_k > 0, \qquad t_0 \equiv 0. \tag{2}$$

Assume that

$$a_{ij} = \mathbf{P}\{\theta_k = j \mid \phi_k = i\} \tag{3}$$

Received April 13, 1959. Presented to the American Mathematical Society, January 20, 1959.