ON FIBER HOMOTOPY EQUIVALENCE

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1. **Introduction.** In [2] A. Dold gives a very useful necessary and sufficient condition for two fiber bundles (over a common polyhedral base with locally compact fibers) to be fiber homotopy equivalent [9]. The objective of this paper is to extend Dold's result to Hurewicz fibrations [5] with no local compactness assumption on the fibers involved. This extension is based on a suitable fiber homotopy extension theorem (§2) and the use of quasi-topologies [7] in certain function spaces. This extension then provides a tool for showing that certain fiber spaces and fiber bundles are fiber homotopy equivalent. As one application we show that any universal bundle [6] over a polyhedron P, whose group is dominated by a CW-complex, is fiber homotopy equivalent to the fiber space of paths emanating from a fixed point of P. Hence it follows that Milnor's universal bundle [6] over P is fiber homotopic to this fiber space of paths.

2. **A fiber homotopy extension theorem.**

2.1 Theorem (FHET). *Let (E, p, X) and (E', p', X) denote Hurewicz fibrations over a common polyhedral base X, and let A denote a subpolyhedron of X. Let $B = (A \times I) \cup (X \times \{0\})$ and $T = (p \times 1)^{-1}(B) = (p^{-1}(A) \times I) \cup (E \times \{0\})$, where $p \times 1: E \times I \to X \times I$. Let $\varphi: T \to E'$ denote a given fiber-preserving partial homotopy, i.e., if $(y, t) \varepsilon T$, $p'\varphi(y, t) = p(y)$. Then, there exists a fiber-preserving homotopy $\Phi: E \times I \to E'$ which extends φ, i.e., $\Phi \mid T = \varphi$ and $p'\Phi(y, t) = p(y)$ for $(y, t) \varepsilon E \times I$.*

Proof. Let U denote an open set in $X \times I$ such that $B \subset U$ and B is a strong deformation retract of U, i.e., there exists a map $H: U \times I \to U$ such that $H_0 = 1$ (= identity map), $H_1(U) \subset B$, and $H(b, t) = b$ for all $b \varepsilon B$, $0 \leq t \leq 1$. Then H induces a corresponding map $\tilde{H}: U \to U^I$ such that for $x \varepsilon U$, $\tilde{H}(x)$ is a path from x to $H_1(x) \varepsilon B$ and $\tilde{H}(b)$, $b \varepsilon B$, is the constant path at b.

Now, let $\pi_1: X \times I \to X$ denote the natural projection map. π_1 then induces $\tilde{\pi}_1: (X \times I)^I \to X^I$. Furthermore, let $V = (p \times 1)^{-1}(U)$. Define a map ψ: $V \to X^I$ by

$$\psi = \tilde{\pi}_1 \circ \tilde{H} \circ (p \times 1) \mid V.$$

We will also make use of the following maps. Let $\pi_2 : E \times I \to E$ $\pi_3 : X \times I \to I$ denote the natural projections and define $\chi: V \to I$ by $\chi = \pi_3 \mid B \circ H_1 \circ (p \times 1) \mid V)$.