THE DENSENESS OF THE EXTREME POINTS OF THE GENERALIZED RESOLVENTS OF A SYMMETRIC OPERATOR

By Richard C. Gilbert

1. **Introduction.** Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$. If $\tilde{A}$ is a self-adjoint extension of A in a possibly larger Hilbert space $\tilde{\mathfrak{H}}$, let $\tilde{R}(\lambda)$ be its resolvent. If P is the operator of orthogonal projection of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$,

$$(1) \qquad R(\lambda) = P\tilde{R}(\lambda) \qquad (I\lambda \neq 0),$$

restricted to $\mathfrak{H}$ is called a *generalized resolvent* of A. We shall say that $\tilde{A}$ *defines* R. The set $\mathcal{R}$ of all generalized resolvents of A is a convex set; see [1; 278]. An element $R \in \mathcal{R}$ is said to be *extreme* if it is impossible to express R in the form $R = \mu_1 R_1 + \mu_2 R_2$, where μ_1, μ_2 are positive real numbers, $\mu_1 + \mu_2 = 1$, and R_1, $R_2 \in \mathcal{R}$, $R_1 \neq R_2$. Let Π be the non-real portion of the complex plane. In [2] it is shown that $\mathcal{R}$ is the closed convex hull of its extreme points, where the topology is that of weak operator convergence, uniformly on compact subsets of Π.

We say that a self-adjoint extension $\tilde{A}$ of A is *finite-dimensional* if dim $(\tilde{\mathfrak{H}} \ominus \mathfrak{H}) < \infty$. If $\tilde{A}$ is a finite-dimensional extension of A, then the generalized resolvent defined by $\tilde{A}$ is an extreme point of $\mathcal{R}$; see [5]. We show in this paper that if A has finite and equal defect indices, then the generalized resolvents of A defined by finite-dimensional extensions are dense in $\mathcal{R}$ in the sense that if $R \in \mathcal{R}$, then there exists a sequence $\{R_n\}$ of generalized resolvents of A, each R_n defined by a finite-dimensional extension of A, such that $\| R_n(\lambda) - R(\lambda) \| \to 0$, uniformly on compact subsets of Π. Thus, the extreme points of $\mathcal{R}$ are dense in $\mathcal{R}$ in the sense of uniform operator convergence, uniformly on compact subsets of Π.

A result similar to this was obtained by I. M. Glazman and P. B. Naiman [3] for the spectral functions of a second order ordinary differential operator on a half-axis.

2. **The generalized resolvent in terms of the parallel projector.** Let A be a closed Hermitian operator in the Hilbert space $\mathfrak{H}$. By Hermitian we mean that $(Af, g) = (f, Ag)$ for all $f, g \in \mathfrak{D}(A)$. (If $\mathfrak{D}(A)$ is dense in $\mathfrak{H}$, then A is symmetric.) If $I\lambda \neq 0$, let $\mathfrak{L}(\lambda) =$ range $(A - \bar{\lambda}E)$ (where E is the identity operator in $\mathfrak{H}$), and let $\mathfrak{M}(\lambda) = \mathfrak{H} \ominus \mathfrak{L}(\lambda)$. $\mathfrak{M}(\lambda)$ is the defect subspace of

Received December 8, 1958. This material is taken in part from the author's doctoral dissertation at the University of California, Los Angeles. The author wishes to express his appreciation to Professor Earl A. Coddington for his counsel, encouragement and technical advice. The work was supported in part by the National Science Foundation, Contract Number G-4226.