EXPANSIONS IN SERIES OF HOMOGENEOUS POLYNOMIAL SOLUTIONS OF THE GENERAL TWO-DIMENSIONAL LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH CONSTANT COEFFICIENTS

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1. **Introduction.** In a previous note [1], we discussed series

$$u(x, y) = \sum_{n=0}^{\infty} P_n(x, y) , \tag{1.1}$$

where $P_n(x, y)$ was a homogeneous polynomial of degree n satisfying the wave equation. Here we assume that $P_n(x, y)$ is a solution of the general linear second order equation with constant coefficients. If $P_n(x, y)$ is not to be identically zero, the equation cannot have first order or zero order terms. (The polynomial solutions of $\partial^2u/\partial x^2 = \partial u/\partial t$ referred to in the earlier note are not homogeneous in x, t but in x, $\sqrt{t}$.) After trivial change of axes the most general equation becomes

$$\frac{\partial^2 u}{\partial x^2} + N \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.2}$$

It is elliptic if $N = M^2 > 0$, hyperbolic if $N = -M^2 < 0$, and parabolic if $N = 0$. We shall show that in the elliptic case the region of convergence is an ellipse $M^2x^2 + y^2 < \rho^2$ plus perhaps certain points outside the ellipse not sufficiently numerous to form a continuum. For example, the series may converge on portions of certain diameters of the ellipse extended beyond its boundary. In the hyperbolic case the region of convergence is a parallelogram $| Mx + y | < \rho_1$, $| Mx - y | < \rho_2$ whose sides are characteristics of (1.2) plus perhaps portions of the extended diagonals. In the parabolic case the region of convergence is an infinite strip, $| y | < \rho$, whose sides are characteristics plus perhaps a portion of a single line through the origin extending beyond the strip. This line may vary from one series to another but may be arbitrary (not a characteristic). Finally, we show that the Taylor (double) series expansion about the origin of the sum $u(x, y)$ of (1.1), converges in the largest parallelogram $M | x | + | y | < \rho$ included in the region of convergence of (1.1) and perhaps on portions of the extended diagonals. In the parabolic case the double series has only two rows and the parallelogram becomes the strip of convergence itself.

2. **Hyperbolic case.** Here our equation is

$$\frac{\partial^2 u}{\partial x^2} - M^2 \frac{\partial^2 u}{\partial y^2} = 0. \tag{2.1}$$

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