RELATIONS AMONG THE MINORS OF A MATRIX WITH DOMINANT PRINCIPAL DIAGONAL

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Introduction. A matrix (a_{ij}) of complex numbers is said to have *dominant principal diagonal* if the absolute value of each principal element a_{ii} exceeds the sum of the absolute values of the non-principal elements on the same row: $| a_{ii} | > \sum_k' | a_{ik} |$. It is known that the determinant of any (square) principal submatrix is not 0, and bounds for this determinant have been given.

In some of the papers [1], [2] which obtained these bounds, it was necessary to know that the determinant of a principal minor exceeds the determinant of an almost-principal minor on the same rows. (An almost-principal minor differs from a principal minor in exactly one column.)

In this note, a set of inequalities is obtained which connects the determinant of a principal minor and the determinants of other minors on the same rows. These inequalities include not only Ostrowski's inequality [1, (14)], but also a more general one which states by how much the determinant of a principal minor exceeds the sum of the determinants of all possible almost-principal minors on the same rows which are obtained by replacing a single fixed column of the almost-principal minor. There is a similar result for almost-almost-principal minors, etc.

It happens that all inequalities of the above type are special cases of a complicated lemma which asserts the truth of a single inequality in which there are involved at the same time almost-principal, almost-almost-principal, etc., minors. (In the special case that all minors are formed from two fixed rows, $p = 2$, the lemma is written out explicity as Corollary 3.) It should not be surprising to those readers who are familiar with inductive proofs that once the lemma has been stated, its proof follows automatically by induction. I have not yet found any essentially simpler inductive proof of a less comprehensive lemma.

Before proceeding to the statement of the theorem it is necessary to establish notational conventions.

By $O_a(b, c)$ is meant the (finite) set of all a-tuples, $t = (t_1, t_2, \cdots, t_a)$ of integers such that $b < t_1 < t_2 < \cdots < t_a \leq C$. For example $O_2(2, 3)$ is the empty set, and $O_2(2, 4)$ is the set $\{(3, 4)\}$.

The theorems of this note concern the determinants of minor matrices of a matrix $A = (a_{ij})$ of p rows and n columns, $n > p$: $1 \leq i \leq p$, $1 \leq j \leq n$. So many minors occur in the proof that it is convenient to have a special notation for them.

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