SOME FUNCTIONS RELATED TO THE BESSEL POLYNOMIALS

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1. Introduction. Krall and Frink [11] introduced the generalized Bessel polynomials, $\{Y_n^{(\alpha)}(x)\}$, as the polynomial solution of the differential equation

$$x^2y''(x) + \{(\alpha + 2)x + 2\}y'(x) = n(n + \alpha + 1)y(x) \tag{1.1}$$

$$(n = 0, 1, 2, 3, \cdots ; \alpha \text{ arbitrary})$$

which takes on the value 1 when $x = 0$. They also proved, among other results, that any three successive polynomials satisfy the second order linear difference equation

$$\begin{aligned}(n + \alpha + 1)(2n + \alpha)Y_{n+1}^{(\alpha)}(x) \\ = (2n + \alpha + 1)\left\{(2n + \alpha)(2n + \alpha + 2)\frac{x}{2} + \alpha\right\}Y_n^{(\alpha)}(x) \\ + n(2n + \alpha + 2)Y_{n-1}^{(\alpha)}(x) \qquad (n \geq 0)\end{aligned} \tag{1.2}$$

with the initial conditions

$$Y_0^{(\alpha)}(x) = 1, \; Y_1^{(\alpha)}(x) = \frac{\alpha + 2}{2}x + 1. \tag{1.3}$$

For references to other authors see [2].

In an earlier paper [2] the writer studied the generalized Bessel polynomials. The present paper is essentially a continuation of [2]. Our main objective here is to construct a sequence of functions $Z_n^{(\alpha)}(x)$ which form a second independent solution of the differential equation (1.1) and a sequence of polynomials $V_n^{(\alpha)}(x)$ which form a second independent solution of the difference equation (1.2). In analogy with the classical orthogonal polynomials, we shall call the $V_n^{(\alpha)}(x)$ the polynomials associated with the $Y_n^{(\alpha)}(x)$.

For the functions $Z_n^{(\alpha)}(x)$ we give various representations, expansion formulas, recurrence relations, generating functions, and multiplication theorems. We also furnish an analytic continuation.

The associated polynomials will be shown to be orthogonal on the unit circle with respect to the weight function

$$\frac{x}{{}_1F_1\left[1; \alpha + 2; -\dfrac{2}{x}\right]}.$$

We further study the location of the zeros of those polynomials and their irreducibility in the rational field in certain special cases. Christoffel-Darboux

Received April 21, 1958.