THE VALUES OF CERTAIN SETS OF MODULES

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1. **Introduction.** In this paper we determine the range of values of sets of modules of simply-connected and doubly-connected domains which satisfy certain conditions. For the doubly-connected domains we consider the module as defined by the method of extremal metric, see [2]. For the simply-connected domains, we consider the reduced module as defined by O. Teichmüller [4]. This reduced module is defined as follows. Let G be a simply-connected domain containing the origin and having at least two boundary points, and let G be mapped conformally onto the unit circle, $|w| = 1$, by the function $w = f(z)$, $z = g(w)$, so that $f(0) = 0$. Then the reduced module of G with respect to the origin is defined to be $M = (1/2\pi) \log |g'(0)|$. If G' is a simply-connected domain containing the point at infinity and having at least two boundary points, let $w = f(z)$ map G' conformally onto the exterior of the unit circle so that $f(\infty) = \infty$. Then if $z = g(w)$ is the inverse of this mapping, the reduced module of G' with respect to the point at infinity is $M' = -(1/2\pi) \log |g'(\infty)|$.

We use the methods of extremal metrics [1] and quadratic differentials [3].

2. Theorem 1. [4]. *Let G' and G'' be any two disjoint simply-connected domains in the w-sphere, such that G' contains $w = 0$, G'' contains the point at infinity and neither contains $w = 1$. Let M' and M'' be the reduced modules of G' and G'', respectively, and let α be a positive parameter. Then the range of values of the pairs (M', M'') is the region in the M', M''-plane which contains the third quadrant and is bounded by the curve defined by the pairs of values*:

$$\left(\frac{1}{2\pi} \log \left| \frac{4\alpha}{\alpha - 1} \left[\frac{\alpha^{\frac{1}{2}} - 1}{\alpha^{\frac{1}{2}} + 1}\right]^{\alpha^{-\frac{1}{2}}} \right|, -\frac{1}{2\pi} \log \left| \frac{\alpha - 1}{4} \left[\frac{\alpha^{\frac{1}{2}} + 1}{\alpha^{\frac{1}{2}} - 1}\right]^{\alpha^{\frac{1}{2}}} \right|\right), \quad \textit{for} \quad \alpha > 1,$$

and

$$\left(\frac{1}{2\pi} \log \left| \frac{4\alpha}{1 - \alpha} \left[\frac{1 - \alpha^{\frac{1}{2}}}{1 + \alpha^{\frac{1}{2}}}\right]^{\alpha^{-\frac{1}{2}}} \right|, -\frac{1}{2\pi} \log \left| \frac{1 - \alpha}{4} \left[\frac{1 + \alpha^{\frac{1}{2}}}{1 - \alpha^{\frac{1}{2}}}\right]^{\alpha^{\frac{1}{2}}} \right|\right), \quad \textit{for} \quad \alpha < 1.$$

Proof. Consider the quadratic differential: $Q(w)\, dw^2 = -(\alpha - w)/w^2(1 - w)\, dw^2$, with $\alpha > 1$. This quadratic differential has a simple pole at $w = 1$, a simple zero at $w = \alpha$, and double poles at $w = 0$ and $w = \infty$. The structure of the trajectories of this quadratic differential is determined as follows. Let $\zeta(w)$ be the function $\zeta = i \int (\alpha - w)^{\frac{1}{2}}/w(1 - w)^{\frac{1}{2}}\, dw$, taking the positive determination of the radicals for $0 < w < 1$, and consider the image of the upper-half

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