AN INVERSE STURM-LIOUVILLE PROBLEM

BY ALLAN D. MARTIN

1. **Introduction.** In recent years there has been some interest in a class of problems in which the traditional relationship between a differential equation and its solutions is reversed. The classical problem in differential equations is concerned with the quest for qualitative or quantitative information about its solutions. In a more recent type of problem certain information, such as the spectrum, is given and the problem is to determine the operators for which the given information is valid. It is a problem of this type which is considered in this paper. Our method is variational.

2. **The focal function.** We consider throughout the second order linear differential operator

$$(r(x)y')' + p(x)y \tag{2.1}$$

and the associated differential equation

$$(r(x)y')' + p(x)y = 0 \tag{2.2}$$

in which r and p are real-valued continuous functions defined on the open interval (a, b) while r is positive there. We term the operator (2.1) *based on* (a, b). Let $c \in (a, b)$ and denote by $B(y, c, \theta)$ the expression

$$y(c) \cos \theta + r(c)y'(c) \sin \theta \tag{2.3}$$

where $0 < \theta < \pi$. Further let $u = u(x, c, \theta)$ as a function of x be a nontrivial solution of (2.2) such that

$$B(u, c, \theta) = 0. \tag{2.4}$$

Then by [1; 223] u has only a finite number of zeros in any closed bounded subinterval of (a, b). Thus let $x = f(c, \theta)$ denote the first zero of u which lies to the right of $x = c$, if it exists. When it does exist, we term it the *θ-focal point* of the line $x = c$. Any other nontrivial solution of (2.2) which satisfies (2.4) is equal to u except for a nonzero multiplicative constant and so its zeros coincide with those of u. Hence f is well defined whenever it is defined. The well known Sturmian theory contains the following theorem.

THEOREM 2.1. *For any x on (a, b), $f(x, \theta)$ is defined for all θ sufficiently small and if θ_0 is any number for which $f(x, \theta_0)$ is defined then $f(x, \theta)$ is also defined for $0 < \theta < \theta_0$. Furthermore f is a strictly increasing function of θ and*

$$\lim_{\theta=0} f(x, \theta) = x. \qquad (a < x < b). \tag{2.5}$$

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