THE REPRESENTATION OF MONADIC BOOLEAN ALGEBRAS

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Introduction. A monadic (Boolean) algebra is a Boolean algebra A together with an operator $\exists$ on A (called an existential quantifier, or, simply, a quantifier) such that

$$\exists 0 = 0,$$

$$p \leq \exists p,$$

$$\exists (p \wedge \exists q) = \exists p \wedge \exists q$$

whenever p and q are in A. A systematic study of monadic algebras appears in [3]; operators such as $\exists$ had occurred before in, for instance, Lewis' studies of modal logic and Tarski's studies of algebraic logic. One motivation for studying monadic algebras is the desire to understand certain aspects of mathematical logic; the connection with logic is also the source of much of the terminology and notation used in the theory. There are, however, many other ways of motivating the study. Monadic algebras arise naturally in, for instance, set theory, pure measure theory, and ergodic theory.

The main purpose of this paper is to present a new proof of the fundamental representation theorem for monadic algebras, much simpler than the original one. At the same time it is pertinent to call attention to some hitherto unnoticed and quite amusing relations among four results: the monadic representation theorem, Sikorski's extension theorem for Boolean homomorphisms [6], Gleason's characterization of "projective" compact spaces [2], and Michael's theorem on cross sections of mappings with a zero-dimensional range [5].

Representation. A typical way to manufacture monadic algebras is this. Let C be an arbitrary Boolean algebra, let k be a positive integer, and let A be a set of ordered k-tuples of elements of C (not necessarily all of them) such that (i) if p and q are in A, then p' and $p \vee q$ are in A (where the Boolean complements and suprema are defined coordinate-wise), and (ii) if p is in A, $p = (p_1, \cdots, p_k)$, and if $\bar{p} = \bigvee_{n=1}^{k} p_n$, then $(\bar{p}, \cdots, \bar{p})$ [k coordinates] is in A. If $\exists p$ is defined to be $(\bar{p}, \cdots, \bar{p})$, then A becomes a monadic algebra.

An infinite generalization of this construction is useful. Let C be an arbitrary Boolean algebra, let K be an arbitrary non-empty set, and let A be a set of functions from K to C such that (i) A is a Boolean algebra with respect to the pointwise Boolean operations, and (ii) if p is in A, then the range of p (a subset of C) has a supremum $\bar{p}$ in C, and the function that takes the value $\bar{p}$ at each point of K is in A. If $\exists p$ is defined to be that function, then A becomes

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