ON THE APPLICATION OF MACMAHON DIAGRAMS TO CERTAIN PROBLEMS IN THE MULTIPLICATION OF MONOMIAL SYMMETRIC FUNCTIONS

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1. **Introduction.** MacMahon [9; 237; 11] has shown how the expansions of monomial symmetric function products can be obtained through the enumeration of certain rectangular arrays composed of zeros and the parts of the monomial factors. In this paper we indicate how these arrays, or MacMahon diagrams, can be used to advantage in the solution of the problems,

(1) To obtain a specific term in the expansion of a product of monomial symmetric functions with numerical parts as a linear sum of such functions.
(2) To obtain the complete expansion of such a product.
(3) To obtain the complete expansions of certain products of monomial symmetric functions with literal parts, these being [A] $(p^i)(q^j)$, [B] $(p^iq^j)(r^k)$, [C] $(p^iq^j)(r^ks^m)$, [D] $(p^iq^jr^k)(s^m)$, [E] $(p^iq^jr^k)(p^mq^n)$, [F] $(p^i)(q^j)(r^k)$.

The expansion of the product [G] (1^p) (n^{e_n}) $(n-1)^{e_{n-1}} \cdots 2^{e_2}1^{e_1})$ is also obtained.

Problems (1) and (2) have, of course, been treated by MacMahon [8], [9], [10] using the basic method of the Hammond operator [5], [9], [11]. In §4, 5 various methods based on MacMahon diagrams, and which have been found to be of value in the solution of these problems, are explained. In §5 it is shown that by a simple change in the formation of the diagrams an effective and relatively rapid method in the solution to Problem (2) can be obtained. This method turns out to be a modification and extension of Cayley's algorithm [1] for the multiplication of two-factor products.

In §6 a detailed solution to one of the cases of expansion [C] is given, and in §7 the results for the expansions [A]–[G] are stated. These were all obtained by aid of the MacMahon diagrams, the solution of §6 being typical.

The expression of various types of symmetric function products as linear sums of symmetric functions is a basic problem in many applications of which we may mention occupancy theory [9], [10], [13], statistics [6], [14], [16], and group representations and characters [4], [7], [12].

A good example of this is seen in the theory of occupancy or distributions as developed by MacMahon [9], [10]. To cite one illustration [10; 22] (related to the present work), the number of distributions of n objects of specification $[m_1m_2 \cdots m_t]$ in r different boxes $B_1, \cdots, B_r$ such that box B_i contains p_i objects of specification $[m_{1i}m_{2i} \cdots m_{k_ii}]$ is given by the coefficient $C_{m_1\cdots m_t}$ of the monomial symmetric function $(m_1 \cdots m_t)$ in the expansion of the product

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