ON THE LOGARITHMIC MEAN CONVERGENCE AND THE REPRESENTATION OF THE REGULAR FUNCTIONS IN THE UNIT CIRCLE

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1. **Introduction.** Let $f(z)$ $(z = re^{i\theta})$ be regular in $|z| < 1$. Put

$$m_p(r) = (1/2\pi) \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \qquad (p > 0, 0 < r < 1),$$

$$m_0(r) = (1/2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \qquad (0 < r < 1).$$

These two mean values of $f(z)$ are the increasing function of r ([2; 159–160]). We define some classes of $f(z)$ as follows:

Definition 1. *If $m_p(r)$ is bounded, we say that $f(z)$ belongs to the class H_p. For brevity, we denote it by $f(z) \in H_p$.*

Definition 2. *If $m_0(r)$ is bounded, we say that $f(z)$ belongs to the class H_0. (In the usual terminology, $f(z)$ is called to be of bounded characteristic.)*

Definition 3. *Let $f(z)$ belong to the class H_0 in $|z| < 1$, and moreover*

$$(1/2\pi) \int_0^{2\pi} |\arg f(re^{i\theta}) - \arg \pi(re^{i\theta})| \, d\theta < M < +\infty \qquad (0 < r < 1),$$

where $\pi(z)$ is the Blaschke's product extended over the zeros of $f(z)$,
M is the constant independent of r.
Then we say that $f(z)$ belongs to the subclass H_0^* of H_0.

By the inequalities:

$$\begin{cases} x^p < 1 + x^q & (0 < p < q, x \geq 0), \\ \log^+ x \leq \max((x^p - 1)/p, 0) & (p > 0, x \geq 0), \end{cases}$$

the next inclusion holds:

$$H_\infty \subset H_q \subset H_p \subset H_0 \qquad (0 < p < q),$$

where H_∞ is the class of bounded regular functions in $|z| < 1$.

F. Riesz ([5], [7; 162], [4; 57 and 61]) has proved that, if $f(z) \in H_p$ $(p > 0)$ in $|z| < 1$, then

$$(1.1) \qquad \begin{cases} \text{(i)} & \lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta}) \quad \text{exists almost everywhere,} \\ \text{(ii)} & f(e^{i\theta}) \in L_p, \\ \text{(iii)} & \lim_{r \to 1} (1/2\pi) \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta = 0. \end{cases}$$

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