SOME STONE SPACES AND RECURSION THEORY

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In this paper certain topological spaces arising in recursion theory are characterized as representation spaces of distributive lattices. Theorems of the type obtained by Areskin [2] for representation spaces of free distributive lattices are extended to these spaces using the topological dual to the imbedding of a distributive lattice in a Boolean algebra (MacNeille [7], Peremans [9]). The partial recursive functionals of recursion theory (Kleene [5], Kuznecov and Trahtenbrot [6], Uspenskiĭ [14]) are identified as continuous maps between representation spaces and thence characterized as dual in a natural sense to a class of homomorphisms between distributive lattices.

1. **Preliminaries.** *A Stone space* [13] is a compact T_0-space X in which: the collection X^* of compact-open sets forms a base for open sets; X^* is closed under finite intersection; whenever A is a subcollection of X^* closed under finite intersection and Y is a closed subset of X such that $a \cap Y$ is non-null for all a in A, then $(\bigcap_{a \varepsilon A} a) \cap Y$ is non-null.

If K is a set, denote by $\mathbf{P}(K)$ the collection of all subsets of K. If $L \subset K$, denote by $\mathbf{U}(L)$ the collection of all subsets of K including L. By the *weak topology* on $\mathbf{P}(K)$ is meant the topology with open base consisting of $\mathbf{U}(L)$ for finite L; weakly closed subspaces of $\mathbf{P}(K)$ coincide with properties of finite character of subsets of K (Birkhoff [3; 42]). An example from logic: suppose M, N are sets and $K = M \times N$. Let $X \subset \mathbf{P}(K)$ consist of all partial functions on M to N; that is, X consists of all functions with domain a subset of M, range a subset of N. Then X is weakly closed.

We assume by definition that a distributive lattice possesses a zero 0 and a unit 1, that a lattice homomorphism preserves 0 and 1, and that a sublattice has the same 0, 1 as the whole lattice. With a distributive lattice A is associated a topological space A^* consisting of all proper prime filters (proper prime dual ideals) of A, regarded as a weak subspace of $\mathbf{P}(A)$.

Theorem 1.1. (Birkhoff-Stone) *Let A be a distributive lattice. Then A^* is a Stone space and there is an isomorphism from A onto A^{**} given by $a \to \hat{a} = [x \varepsilon A^* \mid a \varepsilon x]$. Further, the lattice of ideals of A is isomorphic to the lattice of open sets of A^*, where an ideal I corresponds to the open set $\bigcup_{a \varepsilon I} \hat{a}$. Let X be a Stone space. Then X^* is a distributive lattice of sets and there is a homeomorphism from X onto X^{**} given by $x \to \hat{x} = [a \varepsilon X^* \mid x \varepsilon a]$.*

Call a continuous map between Stone spaces *strongly continuous* if the inverse image of a compact-open set is compact-open.

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